Universal finite functorial semi-norms

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Functorial semi-norms on singular homology measure the "size" of homology classes. A geometrically meaningful example is the ℓ^1-semi-norm. However, the ℓ^1-semi-norm is not universal in the sense that it does not vanish on as few classes as possible. We show that universal finite functorial semi-norms do exist on singular homology on the category of topological spaces that are homotopy equivalent to finite CW-complexes. Our arguments also apply to more general settings of functorial semi-norms.

1 Introduction

A functorial semi-norm on a functor $F\colon C \to \mathsf{Vect}_K$ to vector spaces over a normed field K is a lift of F to a functor $C \to \mathsf{snVect}_K$ to the category of semi-normed vector spaces over K (Definition 2.3). A functorial semi-norm on F is called universal if it vanishes on as few classes as possible among all functorial semi-norms on F (Definition 2.6).

A geometrically meaningful example of a functorial semi-norm is the ℓ^1-semi-norm on singular homology [Gro82], which measures the "size" of homology classes in terms of singular simplices and has applications to rigidity of manifolds [Gro82, BCG91, LS06, CW18]. It is known that the ℓ^1-semi-norm is *not* universal in high degrees [FL19] and it is thus natural to ask whether universal finite functorial semi-norms exist on singular homology [FL19, Question 4.2]. In the present article, we answer this question affirmatively on the category of spaces homotopy equivalent to finite CW-complexes (Corollary 1.2).

More generally, using a suitable diagonalisation technique, we prove the following general existence result (Section 5):

Theorem 1.1. *Let C be a category that admits a skeleton with at most countably many objects. Let K be a normed field and let $F\colon C \to \mathsf{Vect}_K$ be a functor.*

1. *If K is countable and if F maps to Vect_K^ω, then F admits a universal finite functorial semi-norm.*

2. *If F maps to $\mathsf{Vect}_K^{\mathrm{fin}}$, then F admits a universal finite functorial semi-norm.*

Here, $\mathsf{Vect}_K^{\mathrm{fin}}$ and Vect_K^{ω} denote the categories of K-vector spaces of finite or countable dimension, respectively. If the countability assumption on the skeleton is dropped, then in general there does not need to exist a universal finite functorial semi-norm (Section 6).

Instantiating Theorem 1.1 to singular homology, we obtain (Section 5.2):

Corollary 1.2. *Let $d \in \mathbb{N}$ and let K be a normed field (e.g., $\mathbb{R}$). Then the singular homology functor $H_d(\,\cdot\,; K)$ admits a universal finite functorial semi-norm on the category of all topological spaces that are homotopy equivalent to finite CW-complexes.*

In degrees $d \in \{0, 1\}$, it is easy to determine explicit universal finite functorial semi-norms on $H_d(\,\cdot\,; \mathbb{R})$ (Example 2.10). However, the following problems remain open:

Question 1.3. What is the geometric meaning of universal finite functorial semi-norms on singular homology? Are there "nice" examples, at least in degrees 2 and 3?

We reformulate Question 1.3 in more concrete terms in Remark 5.6.

Question 1.4. Let $d \in \mathbb{N}_{\geq 2}$. Does Corollary 1.2 also hold for singular homology on the category of *all* topological spaces?

Remark 1.5 (a comment on sets)**.** As underlying set theory, we use NBG-style sets and classes; this leads to smallness assumptions in some places. Of course, similarly, one could also work in other types of foundations.

Organisation of this article

We start by recalling the notion of (universal) finite functorial semi-norms as well as basic examples and constructions in Section 2. In Section 3, we show that universality is compatible with equivalences of categories. The key construction for universality is presented in Section 4, which allows us to prove the existence results in Theorem 1.1 and Corollary 1.2 in Section 5. Moreover, Section 6 contains an example of a functor that does not admit a universal finite functorial semi-norm.

2 Finite functorial semi-norms

We recall basic notions and examples for finite functorial semi-norms, with a focus on the case of singular homology.

We use the following terminology: Let K be a normed field (e.g., $\mathbb{Q}$ or $\mathbb{R}$ with the standard norm). A *semi-norm* on a K-vector space V is a function $|\,\cdot\,| : V \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ that satisfies

- $|0| = 0$, the

- triangle-inequality, i.e., for all $x, y \in V$ we have $|x + y| \leq |x| + |y|$, and

- homogeneity, i.e., for all $a \in K \setminus \{0\}$ and all $x \in V$ we have $|a \cdot x| = |a| \cdot |x|$

(with the usual conventions regarding ∞). A semi-norm is *finite* if ∞ is not attained. We denote the category of K-vector spaces by Vect_K and the category of semi-normed K-vector spaces with norm non-increasing K-homomorphisms by snVect_K.

Setup 2.1. Let C be a category, let K be a normed field, and let $F\colon C \to \mathsf{Vect}_K$ be a functor.

2.1 Functorial semi-norms

Definition 2.2 (F-element). In the situation of Setup 2.1, an *F-element* is a pair (X, α) where $X \in \mathrm{Ob}(C)$ and $\alpha \in F(X)$. We often suppress X in the notation and simply say that *α is an F-element.*

Definition 2.3 ((finite) functorial semi-norm). We consider the situation of Setup 2.1. A *functorial semi-norm on F* is a lift of F to a functor $\sigma\colon C \to \mathsf{snVect}_K$. Explicitly, the latter consists of a semi-norm $|\cdot|_\sigma$ on $F(X)$ for all objects X of C, such that for all morphisms $f\colon X \to Y$ of C and all $\alpha \in F(X)$ we have

$$\bigl|F(f)(\alpha)\bigr|_\sigma \leq |\alpha|_\sigma.$$

A functorial semi-norm on F is *finite* if $|\cdot|_\sigma$ is finite on $F(X)$ for all X.

Example 2.4 (trivial functorial semi-norm). Every functor in Setup 2.1 admits the *trivial functorial semi-norm*, i.e., the semi-norm that vanishes on every input.

Definition 2.5 (carries). In the situation of Setup 2.1, let σ and τ be functorial semi-norms on F. Then σ *carries* τ if for all F-elements α, we have

$$|\alpha|_\sigma = 0 \implies |\alpha|_\tau = 0.$$

Definition 2.6 (universal finite functorial semi-norm). In the situation of Setup 2.1, a *universal finite functorial semi-norm on F* is a finite functorial semi-norm on F that carries all other finite functorial semi-norms on F.

Remark 2.7. Definition 2.6 is not interesting for the non-finite case, because the functorial semi-norm that is ∞ everywhere, except at 0, is always universal.

Example 2.8 (ℓ^1-semi-norm). Let $d \in \mathbb{N}$. For a topological space X, we set

$$\Bigl|\sum_{j=1}^{N} a_j \cdot \sigma_j\Bigr|_1 := \sum_{j=1}^{N} |a_j|$$

for all reduced singular chains $\sum_{j=1}^{N} a_j \cdot \sigma_j$ in $C_d(X; \mathbb{R})$. The norm $|\cdot|_1$ on $C_d(X; \mathbb{R})$ induces a finite semi-norm $\|\cdot\|_1$ on singular homology $H_d(X; \mathbb{R})$ via

$$\|\alpha\|_1 := \inf\bigl\{|c|_1 \;\big|\; c \in C_d(X; \mathbb{R}) \text{ is a cycle representing } \alpha\bigr\},$$

which is easily seen to be functorial in the sense of Definition 2.3. Hence, we obtain the ℓ^1-*semi-norm* $\| \cdot \|_1$ on $H_d(\cdot \, ; \mathbb{R})$.

An invariant defined in terms of the ℓ^1-semi-norm is the *simplicial volume*, introduced by Gromov [Gro82]: For an oriented closed connected d-dimensional manifold M, the *simplicial volume* $\|M\|$ of M is the ℓ^1-semi-norm $\|M\| := \|[M]_\mathbb{R}\|_1$ of the (real) fundamental class $[M]_\mathbb{R} \in H_d(M; \mathbb{R})$ of M.

The ℓ^1-semi-norm on path-connected spaces also admits other geometric descriptions: It is equivalent (in the sense of semi-norms) to the volume entropy semi-norm [BS19] and to the semi-norm generated by URC-manifolds (Example 2.15).

Example 2.9 (non-universality of the ℓ^1-semi-norm). For each $d \in \{3\} \cup \mathbb{N}_{\geq 5}$ there exists a finite functorial semi-norm on $H_d(\cdot \, ; \mathbb{R})$ that is *not* carried by the ℓ^1-semi-norm [FL19, Theorem 1.2]. However, all finite functorial semi-norms that are multiplicative under finite coverings *are* carried by the ℓ^1-semi-norm [CL15, Proposition 7.11].

Example 2.10 (singular homology in degrees 0 and 1). A direct computation shows that for every topological space X and every $\alpha \in H_0(X; \mathbb{R})$ with $\alpha \neq 0$, we have $\|\alpha\|_1 \neq 0$. In particular, $\| \cdot \|_1$ is a universal finite functorial semi-norm on $H_0(\cdot \, ; \mathbb{R})$.

In contrast, every finite functorial semi-norm $|\cdot|$ on $H_1(\cdot \, ; \mathbb{R})$ vanishes: We first consider the circle. Because S^1 admits a self-map $f \colon S^1 \to S^1$ of degree 2, functoriality gives the estimate $2 \cdot \left|[S^1]_\mathbb{R}\right| = \left|H_1(f; \mathbb{R})[S^1]_\mathbb{R}\right| \leq \left|[S^1]_\mathbb{R}\right|$ and thus

$$\left|[S^1]_\mathbb{R}\right| = 0.$$

For the general case, we observe that the Hurewicz theorem and the universal coefficient theorem show that for every topological space X and every $\alpha \in H_1(X; \mathbb{R})$, there exists $N \in \mathbb{N}$, continuous maps $f_1, \ldots, f_N \in \mathrm{map}(S^1, X)$, and $b_1, \ldots, b_N \in \mathbb{R}$ with

$$\alpha = \sum_{j=1}^N b_j \cdot H_1(f_j; \mathbb{R})([S^1]_\mathbb{R}).$$

Therefore, functoriality and the triangle inequality lead to $|\alpha| \leq \sum_{j=1}^N |b_j| \cdot \left|[S^1]_\mathbb{R}\right| = 0$, as claimed. In particular, the ℓ^1-semi-norm is also universal on $H_1(\cdot \, ; \mathbb{R})$. The principle of representing homology classes by special classes will be discussed in more detail in Section 2.2.

Example 2.11 (representable and countably additive functors). In the situation of Setup 2.1, if $K \in \{\mathbb{Q}, \mathbb{R}\}$ and if the functor F is representable or countably additive, then the trivial functorial semi-norm on F is universal [Löh16, Corollaries 4.1 and 4.5].

2.2 Generating functorial semi-norms

Functorial semi-norms on singular homology lead to estimates for mapping degrees; conversely, properties of mapping degrees can be used to generate functorial semi-norms on singular homology [CL15, Section 4]. This way of "generating functorial semi-norms via special spaces" generalises as follows:

Definition 2.12 (generated semi-norm)**.** In the situation of Setup 2.1, let S be a class of F-elements and let $v \colon S \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.

- An *S-representation* of an F-element (X, α) is a representation of the form

$$\alpha = \sum_{j=1}^{N} b_j \cdot F(f_j)(\alpha_j)$$

  with $N \in \mathbb{N}$, coefficients $b_1, \ldots, b_N \in K$, F-elements $(X_1, \alpha_1), \ldots, (X_N, \alpha_N) \in S$, and morphisms $f_1 \colon X_1 \to X, \ldots, f_N \colon X_N \to X$ in C.

- The *semi-norm* $|\cdot|_v$ *on F generated by v* is defined by: For all F-elements α, we set

$$|\alpha|_v := \inf\left\{ \sum_{j=1}^{N} |b_j| \cdot v(\alpha_j) \;\middle|\; \sum_{j=1}^{N} b_j \cdot F(f_j)(\alpha_j) \text{ is an } S\text{-representation of } \alpha \right\},$$

  with $\inf \emptyset := \infty$.

Proposition 2.13 (generating functorial semi-norms via functions)**.** *In the situation of Definition 2.12, we have:*

1. *The semi-norm $|\cdot|_v$ generated by v is a functorial semi-norm on F.*

2. *For all F-elements α in S, we have $|\alpha|_v \leq v(\alpha)$.*

3. *If $v' \colon S \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a function with $v' \geq v$ (pointwise), then $|\alpha|_{v'} \geq |\alpha|_v$ for all F-elements α. In particular, $|\cdot|_{v'}$ carries $|\cdot|_v$.*

4. *If S contains all F-elements given by a skeleton of C and v does not attain ∞, then $|\cdot|_v$ is finite.*

5. *Let σ be a functorial semi-norm on F and let $v \geq |\cdot|_\sigma$ on S. Then, for all F-elements α, we have $|\alpha|_v \geq |\alpha|_\sigma$.*

Proof. Using functoriality of F, it is easy to see that $|\cdot|_v$ is a functorial semi-norm. Also (3) follows immediately from the definition. For an F-element (X, α), the identity morphism $X \to X$ shows (2). Property (4) is a direct consequence of (2) and the fact that a *functorial* semi-norm is uniquely determined by its restriction to a skeleton. We now prove (5): Let $\sum_{j=1}^{N} b_j \cdot F(f_j)(\alpha_j) = \alpha$ be an S-representation of α. Then

$$|\alpha|_\sigma \leq \sum_{j=1}^{N} |b_j| \cdot \left| F(f_j)(\alpha_j) \right|_\sigma \leq \sum_{j=1}^{N} |b_j| \cdot |\alpha_j|_\sigma \leq \sum_{j=1}^{N} |b_j| \cdot v(\alpha_j)$$

by the triangle inequality, functoriality of σ, and the assumption on v. Taking the infimum over all S-representations of α, we obtain $|\alpha|_v \geq |\alpha|_\sigma$. $\qquad\square$

Remark 2.14 (finiteness of generated semi-norms). Proposition 2.13 (4) only provides a sufficient criterion for $|\cdot|_v$ to be finite. For example, let $d \in \mathbb{N}$ and let us consider the case $F = H_d(\,\cdot\,; \mathbb{R}) \colon \mathsf{Top} \to \mathsf{Vect}_{\mathbb{R}}$. Then, $|\cdot|_v$ is finite whenever S contains and v is finite on enough fundamental classes of manifolds, because rational homology classes can (up to multiplicity) be realised as the push-forward of fundamental classes by a classical result by Thom [Tho54][CL15, Corollary 3.2]. Notably, it is already enough to take the finite coverings of a single URC-manifold in dimension d (Example 2.15).

Example 2.15 (semi-norms generated by URC-manifolds). Let $d \in \mathbb{N}$. An oriented closed connected d-manifold M is a *URC-manifold* (universal realisation of cycles) [Gai13a, p. 1747] if for every topological space X and every $\alpha \in H_d(X; \mathbb{Z})$, there exists a finite-sheeted covering $\overline{M}$ of M, a map $f \in \mathrm{map}(\overline{M}, X)$, and $b \in \mathbb{Z} \setminus \{0\}$ with

$$H_d(f; \mathbb{Z})\big([\overline{M}]_{\mathbb{Z}}\big) = b \cdot \alpha.$$

For example, the point is a URC-manifold in dimension 0, the circle is a URC-manifold in dimension 1, and oriented closed connected surfaces of genus at least 2 are URC-manifolds in dimension 2. Gaifullin showed that (aspherical) URC-manifolds exist in every dimension [Gai13a, Theorem 1.3].
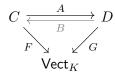
    If M is a URC-manifold in dimension d and S is the class of fundamental classes of all connected finite-sheeted covering manifolds of M, then every homology class in $H_d(\,\cdot\,; \mathbb{R})$ admits an S-representation. Thus, each function $v \colon S \to \mathbb{R}_{\geq 0}$ generates a finite functorial semi-norm on $H_d(\,\cdot\,; \mathbb{R})$ [CL15, Example 7.10].

    If v is given by the covering degree, then $|\cdot|_v$ is equivalent to the ℓ^1-semi-norm on $H_d(\,\cdot\,; \mathbb{R})$ [Gai13b, Theorem 6.1].

3 Universality under equivalence of categories

Universal finite functorial semi-norms are compatible with equivalences of categories (Corollary 3.3). Indeed, a stronger result holds: In Proposition 3.2, we show that universal functorial semi-norms can be transferred along "weak retractions" of categories.

Setup 3.1. Let C and D be categories, let K be a normed field and let $F \colon C \to \mathsf{Vect}_K$ and $G \colon D \to \mathsf{Vect}_K$ be functors. Let $A \colon C \to D$ be a functor such that $G \circ A$ is naturally isomorphic to F.

$$C \underset{B}{\overset{A}{\rightleftarrows}} D$$
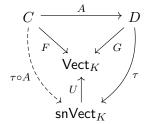$$F \searrow \quad \swarrow G$$
$$\mathsf{Vect}_K$$

Proposition 3.2. *In the situation of Setup 3.1, let $B \colon D \to C$ be a right-inverse of A, i.e., we assume that $A \circ B$ is naturally isomorphic to the identity on D. Then, if F admits a universal functorial semi-norm, so does G.*

    As an immediate consequence, we obtain:

Corollary 3.3. *In the situation of Setup 3.1, assume that $A \colon C \to D$ is an equivalence of categories. Then F admits a universal finite functorial semi-norm if and only if G does.*

Before we give the proof of Proposition 3.2, we make a few remarks about the interplay between functorial semi-norms and natural isomorphisms:

Remark 3.4 (non-strict functorial semi-norms)**.** In the situation of Setup 3.1 and given a functorial semi-norm τ on G, one would like to precompose τ with A to get a functorial semi-norm on F. However, as $G \circ A$ is not necessarily *equal* to F, also $\tau \circ A$ will not necessarily be a strict lift of F, but only up to natural isomorphism. In other words: if $U \colon \mathsf{snVect}_K \to \mathsf{Vect}_K$ denotes the forgetful functor, the right triangle in the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\quad A \quad} & D \\
\end{array}
$$

with F, G to Vect_K; $\tau \circ A$, U, τ to snVect_K

commutes on the nose while the other two only commute up to natural isomorphism.

One possible way to proceed would be to relax the definition of functorial semi-norm: Instead of $U \circ \tau = G$ we only require $U \circ \tau \cong G$, and then the functorial semi-norm consists of τ together with such a natural isomorphism.

This sounds like the correct setting to pursue the categorical view on functorial semi-norms (or formalisation in a proof assistant [Löh22, Chapter 4.1.2]). On the other hand, this setting does not actually increase the pool of functorial semi-norms: Indeed, if $\eta \colon G \Rightarrow U \circ \tau$ is a natural isomorphism, the technique from Remark 3.5 shows how to construct a (strict) functorial semi-norm on G "with the same semi-norms".

Remark 3.5 (pull-back along natural transformation)**.** Let C be a category, let K be a normed field, let $\eta \colon F \Rightarrow F'$ be a natural transformation of functors $C \to \mathsf{Vect}_K$, and let σ be a functorial semi-norm on F'. Then, by naturality of η,

$$
C \to \mathsf{snVect}_K, \quad
\begin{cases}
X \mapsto \big(F(X), (\eta_X)^* |\cdot|_\sigma\big) & \text{on objects} \\
f \mapsto F(f) & \text{on morphisms}
\end{cases}
$$

defines a functorial semi-norm $\eta^* \sigma$ on F.

Proof of Proposition 3.2. First, we first fix some notation: Let $\lambda \colon \mathrm{Id}_D \Rightarrow A \circ B$ and $\psi \colon F \Rightarrow G \circ A$ be a natural isomorphisms. Then $\varphi := \psi^{-1} \circ G(\lambda)$ is a natural isomorphism $G \Rightarrow F \circ B$. We consider the induced functorial semi-norm $\tilde{\sigma} := \varphi^*(\sigma \circ B)$ on G (Remark 3.5).

We show that $\tilde{\sigma}$ is universal for G: Let τ be a finite functorial semi-norm on G. The idea is straightforward: We go to the side of F, compare the result with the universal σ on F, and then derive universality of $\tilde{\sigma}$ on G. However, this involves a round-trip from D over C back to D, and thus we have to take λ into account. More precisely, we proceed as follows:

1. Identifying the goal: Let $(Y, \tilde{\beta})$ be a G-element with $|\tilde{\beta}|_{\tilde{\sigma}} = 0$. We need to show that we also have $|\tilde{\beta}|_{\tau} = 0$.

2. Twisting τ to prepare for the round-trip: We factor in λ by considering the finite functorial semi-norm $\tau_{\lambda} := G(\lambda)^*(\tau \circ A \circ B)$ on G.

3. Going to C: We then use the finite functorial semi-norm $\tilde{\tau} := \psi^*(\tau_{\lambda} \circ A)$ on F.

   Let $\beta := \varphi_Y(\tilde{\beta}) \in F(B(Y))$ be the element corresponding to $\tilde{\beta}$. By construction, we have
   $$|\beta|_{\sigma} = \left|\varphi_Y(\tilde{\beta})\right|_{\sigma} = \left|\varphi_Y(\tilde{\beta})\right|_{\sigma \circ B} = |\tilde{\beta}|_{\varphi^*(\sigma \circ B)} = |\tilde{\beta}|_{\tilde{\sigma}} = 0;$$

   in the second step, we reinterpreted $\varphi_Y(\tilde{\beta})$ as element of $F \circ B(Y)$, so that instead of σ on $B(Y)$ we can equivalently apply $\sigma \circ B$ on Y.

   From universality of σ, we hence obtain $|\beta|_{\tilde{\tau}} = 0$.

4. Translating the result back to D: To keep the notation light, we will not explicitly annotate the objects to which the natural transformations are applied. We compute
   $$0 = |\beta|_{\tilde{\tau}} = |\beta|_{\psi^*(\tau_{\lambda} \circ A)} = \left|\psi(\beta)\right|_{\tau_{\lambda} \circ A} = \left|\psi(\beta)\right|_{\tau_{\lambda}} = \left|\psi(\beta)\right|_{G(\lambda)^*(\tau \circ A \circ B)}$$
   $$= \left|G(\lambda)(\psi(\beta))\right|_{\tau \circ A \circ B} = \left|G(\lambda)(\psi(\beta))\right|_{\tau}.$$

   For every object Z of D, the map $G(\lambda_Z)$ is an isometry with respect to $|\cdot|_{\tau}$ because λ_Z is an isomorphism in D and τ is a functorial semi-norm on G. Therefore, we can continue with
   $$\left|G(\lambda)(\psi(\beta))\right|_{\tau} = \left|\psi(\beta)\right|_{\tau} = \left|\psi(\varphi(\tilde{\beta}))\right|_{\tau} = \left|\psi \circ \psi^{-1} \circ G(\lambda)(\tilde{\beta})\right|_{\tau} = \left|G(\lambda)(\tilde{\beta})\right|_{\tau}$$
   $$= |\tilde{\beta}|_{\tau}.$$

   We conclude that $|\tilde{\beta}|_{\tau} = 0$, as claimed. $\qquad\square$

4 Vanishing loci

In this section, we reformulate the "carries" relation (Definition 2.5) in terms of vanishing loci (Definition 4.2, Remark 4.3).

The vanishing loci provide a convenient language to reason about families of functorial semi-norms and their relations: In Section 4.2, we use a diagonalisation construction on the associated functions to construct a functorial semi-norm that carries countably many given functorial semi-norms (Proposition 4.4 and Corollary 4.5).

Setup 4.1. Let C be a category, let K be a normed field, let $F \colon C \to \mathsf{Vect}_K$ be a functor, and let S be a class of F-elements.

4.1 A reformulation of carrying

Definition 4.2 (vanishing locus)**.** We assume Setup 4.1; let $X \in \mathrm{Ob}(C)$.

- For a functorial semi-norm σ on F, we define the *vanishing locus* of σ on X by

$$N_\sigma(X) := \big\{ \alpha \in F(X) \,\big|\, |\alpha|_\sigma = 0 \big\}.$$

- If C is small, we write $\mathsf{Fsn}(F)$ for the class of all finite functorial semi-norms on F and set

$$N(X) := \bigcap_{\sigma \in \mathsf{Fsn}(F)} N_\sigma(X).$$

- For a function $v \colon S \to \mathbb{R}_{\geq 0}$, we write $N_v(X)$ for $N_{|\cdot|_v}(X)$, where $|\cdot|_v$ is the functorial semi-norm generated by v (Proposition 2.13).

In the situation of the definition, $N_\sigma(X)$ and $N(X)$ are K-subspaces of $F(X)$ and $N(X) \subset N_\sigma(X)$. Furthermore, if we regard $\mathsf{Fsn}(F)$ as the preorder category with respect to the "carries" relation, an initial object of this category is precisely a universal finite functorial semi-norm on F, while the trivial functorial semi-norm is always a terminal one.

Remark 4.3. In the situation of Setup 4.1, let σ, τ be functorial semi-norms on F.

1. Then σ carries τ if and only if

$$\forall_{X \in \mathrm{Ob}(C)} \quad N_\sigma(X) \subset N_\tau(X).$$

2. If C is small, σ is universal on F if and only if it is finite and fulfills

$$\forall_{X \in \mathrm{Ob}(C)} \quad N_\sigma(X) \subset N(X).$$

3. By Proposition 2.13 (5), the functorial semi-norm generated by $S \to \mathbb{R}_{\geq 0}$, $\alpha \mapsto |\alpha|_\sigma$ carries σ, i.e.,

$$\forall_{X \in \mathrm{Ob}(C)} \quad N_{\alpha \mapsto |\alpha|_\sigma}(X) \subset N_\sigma(X).$$

4.2 Carrying a sequence of semi-norms

The main ingredient for the proof of Theorem 1.1 is that we can carry sequences of finite functorial semi-norms generated on a countable class of elements:

Proposition 4.4. *In the situation of Setup 4.1, let S be countable and let $(v_n)_{n \in \mathbb{N}}$ be a sequence of functions $S \to \mathbb{R}_{\geq 0}$. Then there exists a function $v \colon S \to \mathbb{R}_{\geq 0}$ such that $|\cdot|_v$ carries all $(|\cdot|_{v_n})_{n \in \mathbb{N}}$, i.e., with*

$$\forall_{X \in \mathrm{Ob}(C)} \quad N_v(X) \subset \bigcap_{n \in \mathbb{N}} N_{v_n}(X).$$

In particular: If C is small, if every F-element admits an S-representation, and if

$$\forall_{X \in \mathrm{Ob}(C)} \quad \bigcap_{n \in \mathbb{N}} N_{v_n}(X) \subset N(X),$$

then $|\cdot|_v$ is universal for F.

Proof. The second part follows from the first part and the characterisation of universality from Remark 4.3 (2).

We now prove the first part. As indicated by Proposition 2.13 (3), we would like to set "$v := \sup_n v_n$", but of course this might not produce a *finite valued* function. So instead, we choose

$$v \colon S \to \mathbb{R}_{\geq 0}, \ \alpha_n \mapsto \max\big\{v_j(\alpha_n) \,\big|\, j \in \{-1, \dots, m\}\big\},$$

where we fix and implicitly use an enumeration $(X_n, \alpha_n)_{n \in \mathbb{N}}$ of S and where $v_{-1} := 1$.

In order to show that v has the claimed property, we let $m \in \mathbb{N}$ and show that $|\cdot|_v$ carries $|\cdot|_{v_m}$: We introduce the following constants: Let $q_{-1} := 1$, let

$$q_k := \begin{cases} v(\alpha_k) \cdot |\alpha_k|_{v_m}^{-1} & \text{if } |\alpha_k|_{v_m} > 0, \\ 1 & \text{if } |\alpha_k|_{v_m} = 0 \end{cases}$$

for all $k \in \{0, \dots, m\}$, and let

$$Q := \min\big\{q_k \,\big|\, k \in \{-1, \dots, m\}\big\}.$$

By construction, we have that $Q \in (0, 1]$. For every F-element α and every S-representation $\alpha = \sum_{j=1}^{N} b_j \cdot F(f_j)(\alpha_{k_j})$, we can estimate

$$
\begin{aligned}
\sum_{j=1}^{N} |b_j| \cdot v(\alpha_{k_j}) &\geq \sum_{\substack{j \in \{1, \dots, N\} \\ k_j < m}} |b_j| \cdot q_{k_j} \cdot |\alpha_{k_j}|_{v_m} + \sum_{\substack{j \in \{1, \dots, N\} \\ k_j \geq m}} |b_j| \cdot v_m(\alpha_{k_j}) && \text{(definition of } q_{k_j} \text{ and } v) \\
&\geq Q \cdot \sum_{\substack{j \in \{1, \dots, N\} \\ k_j < m}} |b_j| \cdot |\alpha_{k_j}|_{v_m} + \sum_{\substack{j \in \{1, \dots, N\} \\ k_j \geq m}} |b_j| \cdot |\alpha_{k_j}|_{v_m} && \text{(def. of } Q \text{ and P. 2.13 (2))} \\
&\geq Q \cdot \sum_{j=1}^{N} |b_j| \cdot |\alpha_{k_j}|_{v_m} && \text{(because } Q \leq 1) \\
&\geq Q \cdot |\alpha|_{v_m},
\end{aligned}
$$

where the last step follows from applying $|\cdot|_{v_m}$ to the given S-representation of α. By taking the infimum over all such S-representations, we obtain $|\alpha|_v \geq Q \cdot |\alpha|_{v_m}$. As $Q > 0$, we see that $|\cdot|_v$ carries $|\cdot|_{v_m}$ as desired. $\qquad\square$

Corollary 4.5. *In the situation of Setup 4.1, let C be small, let S be countable, and let $T \subset \mathsf{Fsn}(F)$ be countable. Then there exists a functorial semi-norm σ on F such that σ carries all of T, i.e., with*

$$\forall_{X \in \mathrm{Ob}(C)} \quad N_\sigma(X) \subset \bigcap_{\tau \in T} N_\tau(X).$$

In particular: If every F-element admits an S-representation and if

$$\forall_{X \in \mathrm{Ob}(C)} \quad \bigcap_{\tau \in T} N_\tau(X) \subset N(X),$$

then σ is universal for F.

Proof. Again, the second part follows from the first one and Remark 4.3 (2).

We prove the first part of the claim: By Remark 4.3 (3), for each $\tau \in T$, we find a function $v_\tau \colon S \to \mathbb{R}_{\geq 0}$ with

$$\forall_{X \in \mathrm{Ob}(C)} \quad N_{v_\tau}(X) \subset N_\tau(X).$$

We then choose an enumeration of $\{v_\tau \mid \tau \in T\}$ and apply Proposition 4.4. $\square$

5 Existence of universal finite functorial semi-norms

In this section, we prove Theorem 1.1 and Corollary 1.2 on singular homology. We first treat the case of countable fields where a direct enumeration argument applies (Section 5.1). In Section 5.2, we consider functors with range in finite dimensional vector spaces over general normed fields.

In both cases, we use the following observation:

Remark 5.1. By definition, the inclusion functor of a skeleton into the ambient category is an equivalence. Invoking Corollary 3.3, we may equivalently assume that the category itself has only countably many objects.

5.1 The countable case

Proof of Theorem 1.1 (1). We may assume that C itself has only countably many objects (Remark 5.1). Furthermore, by assumption, K and $\dim_K F(X)$ are countable for all objects X of C. Together, we obtain that the class S of *all* F-elements is a countable set. Trivially, all F-elements admit an S-representation.

Let $S' := \{(X, \alpha) \in S \mid \alpha \notin N(X)\}$ and for each $(X, \alpha) \in S'$ let σ_α be a finite functorial semi-norm on F with $\alpha \notin N_{\sigma_\alpha}(X)$.

By construction, for every object Y of C, we have

$$F(Y) \setminus N(Y) \subset \bigcup_{(X, \alpha) \in S'} F(Y) \setminus N_{\sigma_\alpha}(Y).$$

Hence, by De Morgan's laws and Corollary 4.5, there exists a universal functorial semi-norm on F. $\square$

Remark 5.2. In general, it would *not* be enough to have a countable set S with the property that every F-element admits an S-representation. Without the countability assumption on $\mathrm{Ob}(C)$, it might not be possible to control the vanishing locus on all objects by only countably many functorial semi-norms, and thus, the second part of Corollary 4.5 does not apply. A concrete example is given in Section 6.

5.2 The case of finite dimensional range

We prove the second part of Theorem 1.1 and derive Corollary 1.2. As a preparation, we show that we can achieve universality on a single object:

Lemma 5.3. *Let C be a small category, let K be a normed field, and let $F \colon C \to \mathsf{Vect}_K$ be a functor. Let $X \in \mathrm{Ob}(C)$ with $\dim_K F(X) < \infty$. Then there exists a finite functorial semi-norm σ on F with*

$$N_\sigma(X) = N(X).$$

Proof. We proceed inductively, using the following observation:

If $\sigma \in \mathsf{Fsn}(F)$ with $N_\sigma(X) \neq N(X)$, then there exists a $\sigma' \in \mathsf{Fsn}(F)$ with

$$\dim_K N_{\sigma'}(X) < \dim_K N_\sigma(X).$$

Indeed, if $N_\sigma(X) \neq N(X)$, there exists an $\alpha \in N_\sigma(X) \setminus N(X)$. Hence, there is a finite functorial semi-norm τ on F with $|\alpha|_\tau \neq 0$. Then also $\sigma' := \sigma + \tau \in \mathsf{Fsn}(F)$ and α witnesses that

$$N_{\sigma'}(X) \subset N_\sigma(X) \cap N_\tau(X) \subsetneq N_\sigma(X).$$

Because of $\dim_K N_\sigma(X) \leq \dim_K F(X) < \infty$, we obtain $\dim_K N_{\sigma'}(X) < \dim_K N_\sigma(X)$.

For the actual induction, we start with the trivial functorial semi-norm $\sigma := 0$ on F, which satisfies $N_\sigma(X) = F(X)$. We then iteratedly apply the observation above. Because $\dim_K F(X)$ is finite, this will terminate and lead to a finite functorial semi-norm σ on F with $N_\sigma(X) = N(X)$. $\square$

Proof of Theorem 1.1 (2). By Remark 5.1, we may assume without loss of generality, that $\mathrm{Ob}(C)$ is countable. For each $X \in \mathrm{Ob}(C)$, let $(\alpha_i)_{i \in I_X}$ be a finite generating set of the finite-dimensional K-vector space $F(X)$. Then $S := \{(X, \alpha_i) \mid X \in \mathrm{Ob}(C),\ i \in I_X\}$ is countable and every F-element admits an S-representation.

By Lemma 5.3, for each $X \in \mathrm{Ob}(C)$, we find a functorial semi-norm σ_X on F with $N_{\sigma_X}(X) = N(X)$. Therefore, for all $Y \in \mathrm{Ob}(C)$, we have

$$\bigcap_{X \in \mathrm{Ob}(C)} N_{\sigma_X}(Y) \subset N(Y).$$

Applying Corollary 4.5 to the countable set $\{\sigma_X \mid X \in \mathrm{Ob}(C)\}$ thus shows that there exists a universal functorial semi-norm on F. $\square$

Proof of Corollary 1.2. Let T be the category of all topological spaces that are homotopy equivalent to a finite CW-complex; as morphisms in T, we take all continuous maps.

Every functorial semi-norm on $H_d(\,\cdot\,; K)$ is homotopy invariant in the sense that homotopy equivalences induce isometric isomorphisms on $H_d(\,\cdot\,; K)$. Thus, it suffices to show that the functor $F \colon T_{\mathrm{h}} \to \mathsf{Vect}_K$ on the homotopy category T_{h} of T induced by $H_d(\,\cdot\,; K)$ admits a universal finite functorial semi-norm.

As there are only countably many homotopy types of finite CW-complexes (Remark 5.4), the category T_{h} has a skeleton with countably many objects. Moreover, $\dim_K H_d(X; K) < \infty$ for all finite CW-complexes X.

Therefore, the second part of Theorem 1.1 applies and we obtain that F admits a universal finite functorial semi-norm. $\square$

Remark 5.4 (counting CW-complexes). A simple counting argument shows that there are only countably many homeomorphism types of finite simplicial complexes. As every finite CW-complex is homotopy equivalent to a finite simplicial complex [Hat02, Theorem 2C.5], it follows that there are only countably many homotopy types of finite CW-complexes.

In contrast, there are uncountably many homotopy types of countable CW-complexes. Looking at the fundamental group and presentation complexes shows that there are even uncountably many homotopy types of countable 2-dimensional CW-complexes whose 1-skeleton is $S^1 \vee S^1$ (because there are uncountably many isomorphism types of 2-generated groups).

From a constructive point of view, an interesting category of topological spaces with a skeleton that has only countably many objects is the category of recursively enumerable simplicial complexes.

Remark 5.5 (base change). In general, it does not seem to be clear how universal functorial semi-norms behave under base change. For example, if σ is a universal finite functorial semi-norm on a functor F to $\mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fin}}$, then it is not clear whether $\mathbb{R} \otimes_{\mathbb{Q}} \sigma$, defined by the object-wise tensor product with the standard norm on $\mathbb{R}$, is universal for $\mathbb{R} \otimes_{\mathbb{Q}} F$. Indeed, it is a priori not clear how the vanishing loci transform under such base changes.

Remark 5.6 (universal finite functorial semi-norms generated by URC-manifolds). Let $d \in \mathbb{N}$, let M be a URC-manifold, and let S be the class of fundamental classes of all connected finite-sheeted covering manifolds of M (Example 2.15). If $d \geq 2$, then for each $(X, [X]_{\mathbb{R}})$ all covering maps $X \to M$ have the same number of sheets (this can be derived using simplicial volume); we denote this number by $k(X)$. For every $k \in \mathbb{N}$, there are only finitely many homeomorphism types S_k of $(X, [X]_{\mathbb{R}}) \in S$ with $k(X) = k$, as can be seen from the classification of coverings and the fact that the finitely generated group $\pi_1(M)$ contains only a finite number of subgroups of index k.

Let $v \colon S \to \mathbb{R}_{\geq 0}$. We can thus define the modified function

$$\overline{v} \colon S \to \mathbb{R}_{\geq 0},$$
$$(X, \alpha) \mapsto \max_{(X', [X']_{\mathbb{R}}) \in S_{k(X)}} \big| [X']_{\mathbb{R}} \big|_v.$$

By construction $\overline{v} \geq v$ and so $|\cdot|_v$ is carried by $|\cdot|_{\overline{v}}$ (Proposition 2.13 (3)).

Hence, Question 1.3 can be reformulated as follows: How fast does $\overline{v}$ have to grow in the covering degree to ensure that $|\cdot|_{\overline{v}}$ is a universal finite functorial semi-norm on $H_d(\,\cdot\,; \mathbb{R})$ on the category of topological spaces homotopy equivalent to finite CW-complexes? In view of Example 2.9 and Example 2.15, we know that for $d \in \{3\} \cup \mathbb{N}_{\geq 5}$, the growth for universal examples must be faster than linear.

6 Situations without universal finite functorial semi-norms

We give an example of a functor to $\mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fin}}$ that does not admit a universal finite functorial semi-norm (Proposition 6.5). In accordance with Theorem 1.1, the domain category will not admit a skeleton with countably many objects.

Definition 6.1 (the category C). We define a category C by:

- We set $M := (\mathbb{R}_{\geq 1})^{\mathbb{N}}$ and $\mathrm{Ob}(C) := \mathbb{N} \sqcup M$.

- The only non-identity morphisms in C are the morphisms $f_{m,v} \colon m \to v$ with $m \in \mathbb{N}$ and $v \in M$.

Definition 6.2 (the functor F). We define a functor $F \colon C \to \mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fin}}$ as follows:

- For all objects $X \in \mathrm{Ob}(C)$, we set $F(X) := \mathbb{Q}$.

- For $m \in \mathbb{N}$ and $v \in M$, we set

$$F(f_{m,v}) := d_{m,v} \cdot \mathrm{id}_{\mathbb{Q}},$$

where $d_{m,v} := \lceil v(m) \rceil$.

We will show that $F \colon C \to \mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fin}}$ does not admit a universal finite functorial semi-norm. To this end we use the following class to generate functorial semi-norms in the sense of Proposition 2.13:

Definition 6.3 (the class S). For clarity, we denote by 1_X the element $1 \in \mathbb{Q} = F(X)$ for every object $X \in \mathrm{Ob}(C)$. We define

$$S := \big\{ (m, 1_m) \mid m \in \mathbb{N} \big\}$$

and for a function $v \colon \mathbb{N} \to \mathbb{R}_{\geq 0}$, we write $|\cdot|_v := |\cdot|_{(m,1_m) \mapsto v(m)}$ for the generated functorial semi-norm on F.

First we show, that we understand S-representations well enough to compute the generated semi-norms on F:

Lemma 6.4. *In the situation of Definition 6.3, for all $v \colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ and $w \colon \mathbb{N} \to \mathbb{R}_{\geq 1}$, we have*

$$|1_w|_v = \inf_{m \in \mathbb{N}} \frac{1}{d_{m,w}} \cdot v(m).$$

Proof. The S-representations $1_w = \frac{1}{d_{m,w}} \cdot F(f_{m,w})(1_m)$ for $m \in \mathbb{N}$ show that "$\leq$" holds.

Conversely, every S-representation of 1_w is of the form $\sum_{j=1}^{N} b_j \cdot F(f_{m_j,w})(1_{m_j})$ with certain $b_j \in \mathbb{Q}$ and $m_j \in \mathbb{N}$. In particular,

$$1 = |1_w|_{\mathbb{Q}} = \Big| \sum_{j=1}^{N} b_j \cdot d_{m_j,w} \Big|_{\mathbb{Q}} \leq \sum_{j=1}^{N} |b_j|_{\mathbb{Q}} \cdot d_{m_j,w}$$

and so

$$\sum_{j=1}^{N} |b_j|_{\mathbb{Q}} \cdot v(m_j) \geq \sum_{j=1}^{N} |b_j|_{\mathbb{Q}} \cdot d_{m_j,w} \cdot \inf_{m \in \mathbb{N}} \frac{1}{d_{m,w}} \cdot v(m)$$

$$\geq 1 \cdot \inf_{m \in \mathbb{N}} \frac{1}{d_{m,w}} \cdot v(m).$$

Taking the infimum over all S-representations of 1_w finishes the proof. $\qquad\square$

Proposition 6.5. *Let $F \colon C \to \mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fin}}$ be the functor constructed in Definition 6.2 on the category from Definition 6.1. Then, there is* no *universal finite functorial semi-norm on F.*

Proof. Assume for a contradiction that F admits a universal finite functorial semi-norm $|\cdot|$. Let S be the class from Definition 6.3 and let

$$v \colon \mathbb{N} \to \mathbb{R}_{\geq 0}$$
$$m \mapsto |1_m|.$$

Then, v generates a functorial semi-norm $|\cdot|_v$ on F via S (Proposition 2.13, Definition 6.3).

We now consider the function

$$w \colon \mathbb{N} \to \mathbb{R}_{\geq 1}$$
$$m \mapsto m \cdot v(m) + 1$$

and its generated finite functorial semi-norm $|\cdot|_w$ on F.

We show that $|\cdot|_w$ is *not* carried by $|\cdot|$: Let $\alpha := 1_w$. On the one hand, by Lemma 6.4, we obtain

$$|\alpha|_w = \inf_{m \in \mathbb{N}} \frac{1}{d_{m,w}} \cdot w(m) = \inf_{m \in \mathbb{N}} \frac{w(m)}{\lceil w(m) \rceil} \geq \frac{1}{2}.$$

On the other hand, we have (Proposition 2.13 (5) and Lemma 6.4)

$$|\alpha| \leq |\alpha|_v = \inf_{m \in \mathbb{N}} \frac{1}{d_{m,w}} \cdot v(m) = \inf_{m \in \mathbb{N}} \frac{v(m)}{\lceil m \cdot v(m) + 1 \rceil} \leq \inf_{m \in \mathbb{N}_{>0}} \frac{1}{m} = 0.$$

Hence, α witnesses that $|\cdot|_w$ is not carried by $|\cdot|$. $\qquad\square$

It does not seem clear whether this phenomenon could be replicated for the singular homology functor on the category of topological spaces.

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