Transcendental simplicial volumes

Nicolaus Heuer, Clara Löh

July 31, 2020

Abstract

We show that there exist closed manifolds with arbitrarily small transcendental simplicial volumes. Moreover, we exhibit an explicit family of (transcendental) real numbers that are *not* realised as the simplicial volume of a closed manifold.

1 Introduction

The simplicial volume $\|M\| \in \mathbb{R}_{\geq 0}$ is a homotopy invariant of oriented closed connected manifolds M [Mun80, Gro82], namely the ℓ^1-semi-norm of the (singular) $\mathbb{R}$-fundamental class. The set $\mathrm{SV}(d) \subset \mathbb{R}_{\geq 0}$ of simplicial volumes of oriented closed connected d-manifolds is countable and can be determined explicitly in dimensions 1, 2, 3 through classification results [HL19b, Section 2.2]. In these dimensions, simplicial volume has a gap at 0.

In previous work [HL19b], we showed that those are the only dimensions with a gap and that indeed $\mathrm{SV}(d)$ is dense in $\mathbb{R}_{\geq 0}$ for $d \in \mathbb{N}_{\geq 4}$. We also showed that $\mathrm{SV}(4)$ contains $\mathbb{Q}_{\geq 0}$. We now continue these investigations, with a focus on transcendental values.

Theorem A. *For every $\epsilon \in \mathbb{R}_{>0}$, there exists an oriented closed connected 4-manifold M such that*

- *$\|M\|$ is transcendental (over $\mathbb{Q}$) and*
- *$0 < \|M\| < \epsilon$.*

In fact, we provide an explicit sequence of transcendental simplicial volumes of 4-manifolds converging to zero that are linearly independent over the algebraic numbers (Theorem C).

We also give explicit examples of real numbers that are not realised as a simplicial volume:

Theorem B. *Let $d \in \mathbb{N}$ and let $A \subset \mathbb{N}$ be a subset that is recursively enumerable but not recursive. Then*

$$\alpha := \sum_{n \in A} 2^{-n}$$

is transcendental (over $\mathbb{Q}$) and there is no *oriented closed connected d-manifold M with $\|M\| \in \mathbb{R}^c_{>0} \cdot \alpha$, where $\mathbb{R}^c_{>0}$ is the set of positive computable numbers.*

 This work was supported by the CRC 1085 *Higher Invariants* (Universität Regensburg, funded by the DFG).
MSC 2010 classification: 57N65, 57M07, 20J05, 11J86, 03D78

There are many recursively enumerable but non-recursive subsets of $\mathbb{N}$: for example, every encoding of the halting sequence [Cut80, Section 7]; moreover, $1 \in \mathbb{R}^c_{>0}$. Hence, Theorem B provides concrete examples of countably many transcendental numbers that are *not* realised as the simplicial volume of closed manifolds.

We previously explored connections between stable commutator length on finitely presented groups and simplicial volume [HL19a][HL19b, Theorem C/F]; see also Theorem 1.2. Stable commutator length is now well studied in many classes of groups, thanks largely to Calegari and others [Cal09a, Cal09b, Zhu08, CF10, CH19]. Our constructions for the transcendental values of simplicial volumes in Theorems A and C rely on computations by Calegari [Cal09a, Chapter 5].

However, it is unknown which real non-negative numbers are generally realised as the stable commutator length of elements in finitely presented groups. For the larger class of *recursively* presented groups, the set of stable commutator length is known and coincides with the set of right-computable numbers [Heu19]. Thus we ask:

Question 1.1. Does the set of simplicial volumes of oriented closed connected 4-manifolds coincide with the set of non-negative right-computable real numbers?

Proof of Theorem A

Theorem A will follow from the following explicit construction of simplicial volumes:

Theorem C. *There exists a constant $K \in \mathbb{N}_{>0}$ and a sequence $(M_n)_{n\in\mathbb{N}}$ of oriented closed connected 4-manifolds with*

$$\|M_n\| = K \cdot \frac{24 \cdot \arccos(1 - 2^{-n-1})}{\pi}$$

for all $n \in \mathbb{N}$. The numbers $\alpha_n := 24 \cdot \arccos(1 - 2^{-n-1})/\pi$ have the following properties:

1. *We have $\lim_{n\to\infty} \alpha_n = 0$.*
2. *We have $\alpha_0 = 8$ and for each $n \in \mathbb{N}_{>0}$, the number α_n is transcendental (over $\mathbb{Q}$).*
3. *The family $(\alpha_{p-2})_{p\in\mathbb{P}}$ is linearly independent over the field of algebraic numbers; here, $\mathbb{P} \subset \mathbb{N}$ denotes the set of all prime numbers.*

The simplicial volumes constructed in Theorem C will be based on our previous work [HL19b] that allows us to construct 4-manifolds with simplicial volumes prescribed in terms of the stable commutator length of certain finitely presented groups. See Calegari's book [Cal09a] for background on stable commutator length.

Theorem 1.2 ([HL19b, Theorem F])**.** *Let Γ be a finitely presented group that satisfies $H_2(\Gamma;\mathbb{R}) \cong 0$ and let $g \in [\Gamma,\Gamma]$ be an element in the commutator subgroup. Then there exists an oriented closed connected 4-manifold M_g with*

$$\|M_g\| = 48 \cdot \mathrm{scl}_\Gamma\, g.$$

As input for this theorem, we use the following group (whose properties are established in Section 3):

Theorem D. *The central extension* $\widetilde{\Gamma}$ *of* $\mathrm{SL}_2(\mathbb{Z}[1/2])$ *corresponding to the integral Euler class of* $\mathrm{SL}_2(\mathbb{Z}[1/2])$ *is finitely presented. Moreover,* $H_1(\widetilde{\Gamma};\mathbb{Z})$ *is finite and* $H_2(\widetilde{\Gamma};\mathbb{R}) \cong 0$.

It is known that the image of stable commutator length of the central Euler class extension of $\mathrm{SL}_2(\mathbb{Z}[1/2])$ contains arbitrarily small transcendental numbers [Cal09a, Example 5.38]:

Example 1.3. Let $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/2])$ and let $\widetilde{\Gamma}$ denote the central extension of $\mathrm{SL}_2(\mathbb{Z}[1/2])$ corresponding to the integral Euler class of $\mathrm{SL}_2(\mathbb{Z}[1/2])$. In other words, $\widetilde{\Gamma}$ is the pre-image of $\mathrm{SL}_2(\mathbb{Z}[1/2])$ under the canonical projection $\widetilde{\mathrm{SL}_2}(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$, where $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ denotes the universal covering group of $\mathrm{SL}_2(\mathbb{R})$. Then

$$\mathrm{scl}_{\widetilde{\Gamma}}(\widetilde{g}) = \frac{|\mathrm{rot}(\widetilde{g})|}{2}$$

for all $\widetilde{g} \in \widetilde{\Gamma}$, where $\mathrm{rot}\colon \widetilde{\Gamma} \to \mathbb{R}_{\geq 0}$ denotes the rotation number [Cal09a, Example 5.38].

Furthermore, for each $g \in \Gamma$ with $|\mathrm{tr}(g)| \leq 2$, there is a lift $\widetilde{g} \in \widetilde{\Gamma}$ of g such that [Cal09a, p. 145]

$$\mathrm{rot}(\widetilde{g}) = \frac{\arccos(\mathrm{tr}\, g/2)}{\pi}.$$

For $n \in \mathbb{N}_{>0}$, we consider

$$g_n := \begin{pmatrix} 2 & 1 + 2^{-n+1} \\ -1 & -2^{-n} \end{pmatrix} \in \Gamma$$

and let $\widetilde{g}_n \in \widetilde{\Gamma}$ be the associated lift. Then $\lim_{n\to\infty} \mathrm{rot}(\widetilde{g}_n) = 0$ and

$$\mathrm{scl}_{\widetilde{\Gamma}}(\widetilde{g}_n) = \frac{|\mathrm{rot}(\widetilde{g}_n)|}{2} = \frac{\arccos(\mathrm{tr}\, g_n/2)}{2\cdot\pi} = \frac{\arccos(1 - 2^{-n-1})}{2\cdot\pi} = \frac{\alpha_n}{48}.$$

However, a priori, it is not clear that $\widetilde{g}_n$ lies in the commutator subgroup of $\widetilde{\Gamma}$. Because $K := |H_1(\widetilde{\Gamma};\mathbb{Z})|$ is finite (Theorem D), we know that $h_n := \widetilde{g}_n{}^K \in [\widetilde{\Gamma},\widetilde{\Gamma}]$ for all $n \in \mathbb{N}$. Moreover, by construction,

$$\mathrm{scl}_{\widetilde{\Gamma}}(h_n) = K \cdot \mathrm{scl}_{\widetilde{\Gamma}}(\widetilde{g}_n) = K \cdot \frac{\alpha_n}{48}.$$

With these ingredients, we can complete the proof of Theorem C (and thus of Theorem A):

Proof of Theorem C/A. Let $\widetilde{\Gamma}$ be the central Euler class extension of $\mathrm{SL}_2(\mathbb{Z}[1/2])$ and let $(h_n)_{n\in\mathbb{N}}$ and K be as in Example 1.3. Applying Theorem 1.2 to $h_n \in [\widetilde{\Gamma},\widetilde{\Gamma}]$ results in an oriented closed connected 4-manifold M_n with $\|M_n\| = K\cdot\alpha_n$. Hence, $\lim_{n\to\infty} \|M_n\| = K \cdot 24 \cdot \arccos(1)/\pi = 0$. If $n > 0$, then α_n is known to be transcendental (Proposition 2.2). Moreover, Baker's theorem proves the last part of Theorem C (Proposition 2.4). □

Proof of Theorem B

The proof of Theorem B relies on the following simple observation (proved in Section 4, where also the definition of right-computability is recalled):

Theorem E. *Let M be an oriented closed connected manifold. Then $\|M\|$ is a right-computable real number.*

In contrast, the numbers α in Theorem B are *not* right-computable (Proposition 4.3) and thus, in particular, *not* algebraic, because every algebraic number is computable [Eis12, Section 6]. The product of a computable number with a number that is not right-computable is also not right-computable (Section 4.1). Therefore, applying Theorem E proves Theorem B.

Organisation of this article

In Section 2, we prove the transcendence properties of the arccos-terms. In Section 3, we solve the group-theoretic problem for the proof of Theorem D. In Section 4, we prove Theorem E.

Acknowledgements

We would like to thank the anonymous referee for asking questions about the Euler extension (which lead to a simplification of the treatment of Theorem D).

2 Some transcendental numbers

In this section, for $n \in \mathbb{N}_{\geq 0}$, we will investigate the transcendence of the following real numbers

$$\alpha_n := \frac{24 \cdot \arccos(1 - 2^{-n-1})}{\pi}.$$

We will see that $\alpha_0 = 8$ and that α_n is transcendental (over the algebraic numbers) for every $n \geq 1$.

2.1 Transcendence

As a first step, we show that the α_n are transcendental for $n \geq 1$, using Niven's theorem.

Theorem 2.1 (Niven [Niv56, Corollary 3.12])**.** *Let* $\operatorname{trig} \in \{\sin, \cos\}$ *and let* $x \in \mathbb{Q}$ *with* $\operatorname{trig}(\pi \cdot x) \in \mathbb{Q}$. *Then* $\operatorname{trig}(\pi \cdot x) \in \{0, \pm 1/2, \pm 1\}$.

Proposition 2.2. *For every $n \geq 1$, the number α_n is transcendental over $\mathbb{Q}$.*

Proof. A consequence of the Gelfond-Schneider theorem [Lim17, Theorem 1] says that for any real algebraic number x, the expression $\arccos(x)/\pi$ is either rational or transcendental. Thus α_n is either rational or transcendental. *Assume* for a contradiction that α_n were rational. Then, because $\cos(\pi/24 \cdot \alpha_n) = 1 - 2^{-n-1}$ is also rational, by Niven's theorem (Theorem 2.1), we obtain

$$1 - \frac{1}{2^{n+1}} = \cos\Big(\frac{\pi}{24} \cdot \alpha_n\Big) \in \{0, \pm 1/2, \pm 1\}.$$

However, this contradicts the hypothesis that $n \geq 1$. Hence, α_n must be transcendental. □

2.2 Linear independence over the algebraic numbers

We will now refine Proposition 2.2, using Baker's theorem.

Theorem 2.3 (Baker [Bak66])**.** *Let $\Lambda \subset \{\ln(\alpha) \in \mathbb{C} \mid \alpha \text{ algebraic over } \mathbb{Q}\}$ be linearly independent over $\mathbb{Q}$. Then Λ is linearly independent over the field of algebraic numbers.*

Proposition 2.4. *Let $\mathbb{P} \subset \mathbb{N}$ be the set of prime numbers. Then the sequence $(\alpha_{p-2})_{p\in\mathbb{P}}$ is linearly independent over the algebraic numbers.*

For the prime $p = 2$ we compute that $\alpha_{p-2} = \alpha_0 = \frac{24 \arccos(1/2)}{\pi} = 8$, which is rational. Hence, Proposition 2.4 includes a proof that α_{p-2} is transcendental for every odd prime p.

Proof. We will use Baker's Theorem 2.3. Rewriting arccos as

$$\arccos(z) = -i \cdot \ln\big(i \cdot z + \sqrt{1 - z^2}\big),$$

we see that

$$\alpha_{p-2} = \frac{24 \cdot \arccos(1 - 2^{-p+1})}{\pi} = \frac{-24 \cdot i}{\pi} \cdot \ln(\gamma_p),$$

where

$$\gamma_p := i \cdot \frac{2^{p-1} - 1}{2^{p-1}} + \frac{1}{2^{p-1}} \cdot \sqrt{2^p - 1}.$$

We will show in Claim 2.8 that for every finite set $\{p_1, \dots, p_k\}$ of distinct primes the family $\{\ln(\gamma_{p_j})\}_{j\in\{1,\dots,k\}}$ is linearly independent over $\mathbb{Q}$. As α_{p-2} is a uniform rescaling of $\ln(\gamma_p)$, this will imply by using Baker's Theorem that this family is also linearly independent over the algebraic numbers.

We will show the linear independence of $\{\ln(\gamma_{p_j})\}_{j\in\{1,\dots,k\}}$ over $\mathbb{Q}$ in several steps:

Claim 2.5. *Let $(m_k)_{k\in\mathbb{N}}$ be a sequence of pairwise coprime positive integers. Then, for every $k \in \mathbb{N}_{\geq 2}$, we have that*

$$\sqrt{m_k} \notin \mathbb{Q}[i, \sqrt{m_1}, \dots, \sqrt{m_{k-1}}].$$

Proof. This follows from a classical result of Besicovitch [Bes40]. □

Claim 2.6. *Let $\{p_1, \dots, p_k\}$ be a finite set of distinct primes. Then*

$$\sqrt{2^{p_k} - 1} \notin \mathbb{Q}[i, \sqrt{2^{p_1} - 1}, \sqrt{2^{p_2} - 1}, \dots, \sqrt{2^{p_{k-1}} - 1}]$$

Proof. For all primes $p, q \in \mathbb{N}$ with $p \neq q$, the Mersenne numbers $2^p - 1$ and $2^q - 1$ are coprime. We may conclude using the previous claim. □

Claim 2.7. *Let $\{p_1, \dots, p_k\}$ be a finite set of distinct primes and let $n \in \mathbb{N}_{>0}$. Then*

$$\gamma_{p_k}^n \notin \mathbb{Q}[i, \sqrt{2^{p_{k-1}} - 1}, \sqrt{2^{p_{k-2}} - 1}, \dots, \sqrt{2^{p_1} - 1}].$$

Proof. We compute that

$$\begin{aligned}\gamma_{p_k}^n &= \Big(i \cdot \frac{2^{p_k-1}-1}{2^{p_k-1}} + \frac{1}{2^{p_k-1}} \cdot \sqrt{2^{p_k}-1}\Big)^n \\ &= \frac{1}{2^{n(p_k-1)}} \cdot \sum_{j=0}^{n} \binom{n}{j} \cdot i^{n-j} \cdot (2^{p_k-1}-1)^{n-j} \cdot (2^{p_k}-1)^{\frac{j}{2}}.\end{aligned}$$

We see that the terms contributing to $\sqrt{2^{p_k}-1}$ are the terms where j is odd and that there exist $q_1, q_2 \in \mathbb{Q}$ with

$$\gamma_{p_k}^n = i^n \cdot (q_1 + q_2 \cdot i \cdot \sqrt{2^{p_k}-1}).$$

Assume for a contradiction that q_2 were zero. Then $\gamma_{p_k} \in \mathbb{Q} \cup i \cdot \mathbb{Q}$ and as $|\gamma_{p_k}| = 1$ we obtain $\gamma_{p_k}^n \in \{\pm 1, \pm i\}$. In particular, γ_{p_k} is a root of unity. Therefore, there exists an $x \in \mathbb{Q}$ with

$$\gamma_{p_k} = \cos(2\pi \cdot x) + i \cdot \sin(2\pi \cdot x).$$

According to Niven's Theorem 2.1, by comparing with the definition of γ_{p_k}, we see that $\frac{2^{p_k-1}}{2^{p_k}} \in \{0, \frac{1}{2}, 1\}$. But if p_k is a prime, then this never happens. Hence, q_2 is non-zero, and so $\gamma_{p_k}^n \notin \mathbb{Q}[i, \sqrt{2^{p_1}-1}, \dots, \sqrt{2^{p_{k-1}}-1}]$ by Claim 2.6. □

Claim 2.8. *Let $\{p_1, \dots, p_k\}$ be a finite set of distinct primes. Then the corresponding family $\{\ln(\gamma_{p_j})\}_{j \in \{1,\dots,k\}}$ is linearly independent over $\mathbb{Q}$.*

Proof. Assume for a contradiction that this family were linearly dependent over $\mathbb{Q}$, whence over $\mathbb{Z}$. Thus, there are integers $n_i \in \mathbb{Z}$, not all zero, such that

$$\ln(\gamma_{p_1}^{n_1} \cdots \gamma_{p_k}^{n_k}) = n_1 \cdot \ln(\gamma_{p_1}) + \cdots + n_k \cdot \ln(\gamma_{p_k}) = 0.$$

Without loss of generality we may assume that $n_k > 0$. Hence,

$$\gamma_{p_1}^{n_1} \cdots \gamma_{p_k}^{n_k} \in \{1 + m \cdot 2\pi i \mid m \in \mathbb{Z}\}.$$

The left-hand side is algebraic over $\mathbb{Q}$, but the right-hand side is only algebraic if $m = 0$. Thus, we conclude that $\gamma_{p_1}^{n_1} \cdots \gamma_{p_k}^{n_k} = 1$; in other words,

$$\gamma_{p_k}^{n_k} = \gamma_{p_1}^{-n_1} \cdots \gamma_{p_{k-1}}^{-n_{k-1}}.$$

Moreover, by construction, $\gamma_{p_1}^{-n_1} \cdots \gamma_{p_{k-1}}^{-n_{k-1}} \in \mathbb{Q}[i, \sqrt{2^{p_1}-1}, \dots, \sqrt{2^{p_{k-1}}-1}]$. However, this contradicts Claim 2.7. Thus, $\ln(\gamma_{p_1}), \dots, \ln(\gamma_{p_k})$ are linearly independent over $\mathbb{Q}$. □

This finishes the proof of Proposition 2.4. □

3 Solving the group-theoretic problem

As the basic building block for our constructions we pick $\mathrm{SL}_2(\mathbb{Z}[1/2])$ because its low-degree (co)homology, its second bounded cohomology, and its quasi-morphisms are already known to basically have the right structure.

3.1 Basic properties of $\mathrm{SL}_2(\mathbb{Z}[1/2])$

We collect basic properties of $\mathrm{SL}_2(\mathbb{Z}[1/2])$ needed in the sequel; further information on the (bounded) Euler class for circle actions can be found in the literature [BFH16, Ghy87].

Proposition 3.1 (low-degree (co)homology of $\mathrm{SL}_2(\mathbb{Z}[1/2])$)**.**

1. *The group* $\mathrm{SL}_2(\mathbb{Z}[1/2])$ *is finitely presented.*
2. *The group* $H_1(\mathrm{SL}_2(\mathbb{Z}[1/2]);\mathbb{Z})$ *is finite (and non-trivial).*
3. *The group* $\mathrm{SL}_2(\mathbb{Z}[1/2])$ *does* not *admit any non-trivial quasi-morphisms.*
4. *We have* $H_b^2(\mathrm{SL}_2(\mathbb{Z}[1/2]);\mathbb{R}) \cong \mathbb{R}$ *and the bounded Euler class* ${}^{\mathrm{SL}_2(\mathbb{Z}[1/2])}\mathrm{eu}_b^{\mathbb{R}}$ *is a generator.*
5. *The evaluation map* $\langle {}^{\mathrm{SL}_2(\mathbb{Z}[1/2])}\mathrm{eu}^{\mathbb{Z}}, \,\cdot\, \rangle \colon H_2(\mathrm{SL}_2(\mathbb{Z}[1/2]);\mathbb{Z}) \to \mathbb{Z}$ *has finite kernel and finite cokernel.*

Proof. Ad 1. The group $\mathrm{SL}_2(\mathbb{Z}[1/2])$ can be written as an amalgamated free product of the form

$$\mathrm{SL}_2(\mathbb{Z}[1/2]) \cong \mathrm{SL}_2(\mathbb{Z}) *_{\Gamma_0(2)} \mathrm{SL}_2(\mathbb{Z}),$$

where $\Gamma_0(2)$ is the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of those matrices whose lower left entry is divisible by 2; this leads to an explicit finite presentation [Ser03, p. 81].

Ad 2. In particular, one obtains that $H_1(\mathrm{SL}_2(\mathbb{Z}[1/2]);\mathbb{Z}) \cong \mathbb{Z}/3$ is finite [AN98, Proposition 3.1]. (Moreover, applying the Mayer-Vietoris sequence to the decomposition in the proof of the first part allows to compute the cohomology $H^*(\mathrm{SL}_2(\mathbb{Z}[1/2]);\mathbb{Z})$ [AN98].)

Ad 3. This is one of many examples of groups acting on the circle with this property [Cal09a, Example 5.38].

Ad 4. This is a result of Burger and Monod: The inclusion $\mathrm{SL}_2(\mathbb{Z}[1/2]) \to \mathrm{SL}_2(\mathbb{R})$ induces an isomorphism $H_{cb}^2(\mathrm{SL}_2(\mathbb{R});\mathbb{R}) \to H_b^2(\mathrm{SL}_2(\mathbb{Z}[1/2]);\mathbb{R})$ [BM02a, Corollary 24][BM19, Corollary 4]. Moreover, $H_{cb}^2(\mathrm{SL}_2(\mathbb{R});\mathbb{R}) \cong \mathbb{R}$, generated by the bounded Euler class [BM02b].

Ad 5. We abbreviate $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/2])$. Because Γ is finitely presented, $H_2(\Gamma;\mathbb{Z})$ is a finitely generated Abelian group [Bro94, II.5]. Moreover, it has been computed that $H_2(\Gamma;\mathbb{Q}) \cong \mathbb{Q}$ [Mos80, Proposition 2.2]. Hence, $H_2(\Gamma;\mathbb{Z})$ is virtually $\mathbb{Z}$ and it suffices to show that the evaluation $\langle {}^{\Gamma}\mathrm{eu}^{\mathbb{Z}}, \,\cdot\, \rangle \colon H_2(\Gamma;\mathbb{Z}) \to \mathbb{Z}$ is non-trivial.

As the space $Q(\Gamma)$ of quasi-morphisms (modulo trivial quasi-morphisms) is trivial, the comparison map $c_\Gamma \colon H_b^2(\Gamma;\mathbb{R}) \to H^2(\Gamma;\mathbb{R})$ is injective [Cal09a, Theorem 2.50]. In particular, ${}^{\Gamma}\mathrm{eu}^{\mathbb{R}} = c_\Gamma({}^{\Gamma}\mathrm{eu}_b^{\mathbb{R}})$ is non-trivial in $H^2(\Gamma;\mathbb{R})$. Therefore, by the universal coefficient theorem, also the evaluation $\langle {}^{\Gamma}\mathrm{eu}^{\mathbb{Z}}, \,\cdot\, \rangle \colon H_2(\Gamma;\mathbb{Z}) \to \mathbb{Z}$ associated with the integral Euler class ${}^{\Gamma}\mathrm{eu}^{\mathbb{Z}} \in H^2(\Gamma;\mathbb{Z})$ is non-trivial. ☐

3.2 Imitating the universal central extension

If Γ is a perfect group, then its universal central extension E is a perfect group that satisfies $H_2(E;\mathbb{R}) \cong 0$. The universal central extension of Γ can be constructed as the central extension corresponding to the cohomology class φ

in $H^2(\Gamma; H_2(\Gamma;\mathbb{Z}))$ whose evaluation map $\langle \varphi, \,\cdot\, \rangle \colon H_2(\Gamma;\mathbb{Z}) \to H_2(\Gamma;\mathbb{Z})$ is the identity map. Moreover, we may compute the quasimorphisms on E from $H^2_b(\Gamma;\mathbb{R})$, which in turn allows us to compute the stable commutator length on E using Bavard's Duality Theorem [HL19b, Section 5]. The group $\mathrm{SL}_2(\mathbb{Z}[1/2])$ is not perfect, thus it does not have a universal central extension. Instead, we will choose a central extension of $\mathrm{SL}_2(\mathbb{Z}[1/2])$ that is able to play the same role in our context.

Proposition 3.2. *Let Γ be a finitely presented group with finite $H_1(\Gamma;\mathbb{Z})$, let A be a finitely generated Abelian group, and let E be a central extension group of Γ that corresponds to a class $\varphi \in H^2(\Gamma; H)$ such that the evaluation map $\langle \varphi, \,\cdot\, \rangle \colon H_2(\Gamma;\mathbb{Z}) \to A$ has finite kernel and finite cokernel. Then:*

1. *The group E is finitely presented.*
2. *We have $H_1(E;\mathbb{R}) \cong 0$ and $H_2(E;\mathbb{R}) \cong 0$.*

Proof. The central extension group E fits into a short exact sequence of the form $1 \longrightarrow A \longrightarrow E \longrightarrow \Gamma \longrightarrow 1$.

Ad 1. Because A is finitely generated, the central extension group E of Γ by A is also finitely presented.

Ad 2. Because the extension is central, we have the associated exact sequence

$$H_1(E;\mathbb{Z}) \otimes_\mathbb{Z} A \longrightarrow H_2(E;\mathbb{Z}) \longrightarrow H_2(\Gamma;\mathbb{Z}) \xrightarrow{\beta} A \longrightarrow H_1(E;\mathbb{Z}) \longrightarrow H_1(\Gamma;\mathbb{Z}) \longrightarrow 0$$

by Eckmann, Hilton, and Stammbach [EHS72, (1.4) and Theorem 2.2], where

$$\begin{aligned} \beta \colon H_2(\Gamma;\mathbb{Z}) &\to A \\ \alpha &\mapsto \langle \varphi, \alpha \rangle. \end{aligned}$$

By assumption, β has finite cokernel and $H_1(\Gamma;\mathbb{Z})$ is finite. Hence, $H_1(E;\mathbb{Z})$ is finite and therefore also the left-most term $H_1(E;\mathbb{Z}) \otimes_\mathbb{Z} A$ is finite. As β has finite kernel, this implies that $H_2(E;\mathbb{Z})$ is finite. Applying the universal coefficient theorem, shows that $H_2(E;\mathbb{R}) \cong H_2(E;\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{R} \cong 0$. ☐

With these preparations, we can now give a proof of Theorem D:

Proof of Theorem D. We only need to combine Propositions 3.1 and 3.2. As $\widetilde{\Gamma}$ is finitely generated, $H_1(\widetilde{\Gamma};\mathbb{R}) \cong 0$ implies that $H_1(\widetilde{\Gamma};\mathbb{Z})$ is finite. ☐

3.3 More on almost universal extensions

Let us mention that the same procedure as in the previous proofs also works in other, similar, situations:

Setup 3.3. Let Γ be a group with a given orientation preserving continuous action on S^1 with the following properties:

- The group Γ is finitely presented.
- The group $H_1(\Gamma;\mathbb{Z})$ is finite.
- The group Γ does *not* admit any non-trivial quasi-morphisms.

- We have $H_b^2(\Gamma;\mathbb{R}) \cong \mathbb{R}$ and the bounded Euler class ${}^\Gamma\mathrm{eu}_b^{\mathbb{R}}$ is a generator.

In this situation, we denote the central extension group of Γ associated with the Euler class ${}^\Gamma\mathrm{eu}^{\mathbb{Z}} \in H^2(\Gamma;\mathbb{Z})$ by $\widetilde{\Gamma}$.

We have already seen in the previous propositions that $\mathrm{SL}_2(\mathbb{Z}[1/2])$ fits into this setup. Another prominent example is Thompson's group T, which is even perfect; the condition on H_b^2 follows from explicit cohomological computations [HL19b, Proposition 5.6], based on calculations by Ghys and Sergiescu [GS87].

Proposition 3.4. *Let Γ be as in Setup 3.3. Then:*

1. *The evaluation map $\langle {}^\Gamma\mathrm{eu}^{\mathbb{Z}}, \,\cdot\, \rangle \colon H_2(\Gamma;\mathbb{Z}) \to \mathbb{Z}$ is non-trivial.*
2. *Let $H := H_2(\Gamma;\mathbb{Z})$, let $m \in \mathbb{N}_{>0}$ be a generator of $\mathrm{im}\langle {}^\Gamma\mathrm{eu}^{\mathbb{Z}}, \,\cdot\, \rangle \subset \mathbb{Z}$ (which is non-zero by the first part), and let $\epsilon := 1/m \cdot \langle {}^\Gamma\mathrm{eu}^{\mathbb{Z}}, \,\cdot\, \rangle \colon H \to \mathbb{Z}$. Then there exists a $\varphi \in H^2(\Gamma;\mathbb{Z})$ with*
$$H^2(\mathrm{id}_\Gamma;\epsilon)(\varphi) = {}^\Gamma\mathrm{eu}^{\mathbb{Z}} \quad \text{and} \quad \langle \varphi, \,\cdot\, \rangle = m \cdot \mathrm{id}_H\,.$$
3. *Let E be the central extension group of Γ associated with φ. Then there exists an epimorphism $\psi\colon E \to \widetilde{\Gamma}$ with $\psi|_H = \epsilon\colon H \to \mathbb{Z}$ and $\ker\psi \subset H$.*

Proof. Ad 1. This is the same universal coefficient theorem argument as in the last part of (the proof of) Proposition 3.1.

Ad 2. By the naturality of the short exact sequence in the universal coefficient theorem, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}\big(H_1(\Gamma;\mathbb{Z}),H\big) & \longrightarrow & H^2(\Gamma;H) & \xrightarrow{\varphi\mapsto\langle\varphi,\,\cdot\,\rangle} & \mathrm{Hom}_{\mathbb{Z}}(H,H) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Ext}^1(\mathrm{id};\epsilon)} & & \downarrow{\scriptstyle H^2(\mathrm{id}_\Gamma;\epsilon)} & & \downarrow{\scriptstyle f\mapsto\epsilon\circ f} & & \\
0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}\big(H_1(\Gamma;\mathbb{Z}),\mathbb{Z}\big) & \longrightarrow & H^2(\Gamma;\mathbb{Z}) & \xrightarrow[\varphi\mapsto\langle\varphi,\,\cdot\,\rangle]{} & \mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

The left vertical arrow is an epimorphism because ϵ is an epimorphism and the exactness properties of Ext over the principal ideal domain $\mathbb{Z}$. Moreover, the right vertical arrow maps $m \cdot \mathrm{id}_H$ to $m \cdot \epsilon = \langle {}^\Gamma\mathrm{eu}^{\mathbb{Z}}, \,\cdot\, \rangle$. A short diagram chase therefore proves the existence of the desired class $\varphi \in H^2(\Gamma;H)$ (e.g., using the four lemma [ML95, Lemma I.3.2]).

Ad 3. Because the extension classes are related via $H^2(\mathrm{id}_\Gamma;\epsilon)(\varphi) = {}^\Gamma\mathrm{eu}^{\mathbb{Z}}$, there exists a group homomorphism $\psi\colon E \to \widetilde{\Gamma}$ with $\psi|_H = \epsilon$ that induces the identity on Γ:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \epsilon} & & \uparrow{\scriptstyle \psi} & & \| & & \\
1 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array}$$

As $\epsilon\colon H \to \mathbb{Z}$ is an epimorphism also $\psi\colon E \to \widetilde{\Gamma}$ is an epimorphism. By construction, $\ker\psi \subset H$. ◻

Corollary 3.5. *Let Γ be as in Setup 3.3, let $H := H_2(\Gamma;\mathbb{Z})$, and let E be the central extension group of Γ associated with the class $\varphi \in H^2(\Gamma;H)$ of Proposition 3.4. Then:*

1. *The group E is finitely presented and $H_2(E;\mathbb{R}) \cong 0$.*
2. *The epimorphism $\psi\colon E \to \widetilde{\Gamma}$ of Proposition 3.4 induces an isomorphism*
$$\begin{aligned} Q(\psi)\colon Q(\widetilde{\Gamma}) &\to Q(E) \\ [f] &\mapsto [f \circ \psi] \end{aligned}$$
and both spaces are one-dimensional. Here, Q denotes the space of quasi-morphisms modulo trivial quasi-morphisms.
3. *In particular, $\operatorname{scl}_E([E,E]) = \operatorname{scl}_{\widetilde{\Gamma}}([\widetilde{\Gamma},\widetilde{\Gamma}])$ as subsets of $\mathbb{R}$.*

Proof. Ad 1. This follows directly from Proposition 3.2.

Ad 2. We will use bounded cohomology in degree 2 to derive the statement on quasi-morphisms; we consider the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & Q(\widetilde{\Gamma}) & \xrightarrow{\delta} & H_b^2(\widetilde{\Gamma};\mathbb{R}) & \xrightarrow{c^2_{\widetilde{\Gamma}}} & H^2(\widetilde{\Gamma};\mathbb{R}) \\ & \downarrow{\scriptstyle Q(\psi)} & & \downarrow{\scriptstyle H_b^2(\psi;\mathbb{R})} & & \downarrow{\scriptstyle H^2(\psi;\mathbb{R})} \\ 0 \longrightarrow & Q(E) & \xrightarrow[\delta]{} & H_b^2(E;\mathbb{R}) & \xrightarrow[c^2_E]{} & H^2(E;\mathbb{R}) \end{array}$$

with exact rows.

By construction, the kernel of the epimorphism $\psi\colon E \to \widetilde{\Gamma}$ lies in the Abelian group H and thus is amenable. By the mapping theorem in bounded cohomology [Gro82, p. 40][Iva85, Theorem 4.3], the induced map $H_b^2(\psi;\mathbb{R})\colon H_b^2(\widetilde{\Gamma};\mathbb{R}) \to H_b^2(E;\mathbb{R})$ is an isomorphism.

Because $H_2(E;\mathbb{R}) \cong 0$, we also have $H^2(E;\mathbb{R}) \cong 0$. Therefore, $\delta\colon Q(E) \to H_b^2(E;\mathbb{R})$ is an isomorphism.

We now show that also $\delta\colon Q(\widetilde{\Gamma}) \to H_b^2(\widetilde{\Gamma};\mathbb{R})$ is an isomorphism: By the mapping theorem in bounded cohomology, the extension projection $\widetilde{\pi}\colon \widetilde{\Gamma} \to \Gamma$ induces an isomorphism $H_b^2(\widetilde{\pi};\mathbb{R})\colon H_b^2(\Gamma;\mathbb{R}) \to H_b^2(\widetilde{\Gamma};\mathbb{R})$. As $H_b^2(\Gamma;\mathbb{R})$ is generated by the bounded Euler class, also $H_b^2(\widetilde{\Gamma};\mathbb{R})$ is one-dimensional and generated by
$$\widetilde{\mathrm{eu}} := H_b^2(\widetilde{\pi};\mathbb{R})({}^\Gamma\mathrm{eu}_b^{\mathbb{R}}).$$
By naturality of the comparison map, we obtain that
$$c^2_{\widetilde{\Gamma}}(\widetilde{\mathrm{eu}}) = H^2(\widetilde{\pi};\mathbb{R})({}^\Gamma\mathrm{eu}^{\mathbb{R}}).$$
By construction of the central Euler class extension $\widetilde{\Gamma}$, we have $H^2(\widetilde{\pi};\mathbb{Z})({}^\Gamma\mathrm{eu}^{\mathbb{Z}}) = 0 \in H^2(\widetilde{\Gamma};\mathbb{Z})$. Therefore, $H^2(\widetilde{\pi};\mathbb{R})({}^\Gamma\mathrm{eu}^{\mathbb{R}}) = 0$ and so $c^2_{\widetilde{\Gamma}}(\widetilde{\mathrm{eu}}) = 0$. This shows that $\delta\colon Q(\widetilde{\Gamma}) \to H_b^2(\widetilde{\Gamma};\mathbb{R})$ is an isomorphism.

Now commutativity of the left square in the diagram above shows that $Q(\psi)\colon Q(\widetilde{\Gamma}) \to Q(E)$ is an isomorphism.

Ad 3. Let $[f] \in Q(\widetilde{\Gamma}) \cong \mathbb{R}$ be a homogeneous generator, which exists by the second part; then $[f \circ \psi]$ is a homogeneous generator of $Q(E)$. Bavard duality [Bav91][Cal09a, Theorem 2.70] implies that for all $g \in [E, E]$, we have

$$\mathrm{scl}_E(g) = \frac{\bigl|f \circ \psi(g)\bigr|}{2 \cdot D_E(f \circ \psi)} = \frac{\bigl|f(\psi(g))\bigr|}{2 \cdot D_{\widetilde{\Gamma}}(f)} = \mathrm{scl}_{\widetilde{\Gamma}}\bigl(\psi(g)\bigr);$$

the defects in the denominators are equal because ψ is an epimorphism. Again, because ψ is an epimorphism, we conclude that scl_E and $\mathrm{scl}_{\widetilde{\Gamma}}$ have the same image in $\mathbb{R}$. □

4 Right-computability of simplicial volumes

We now turn to right-computability of the numbers occuring as simplicial volumes. After recalling basic terminology in Section 4.1, we will prove Theorem E in Section 4.2.

4.1 Right-computability

We use the following version of (right-)computability of real numbers, which is formulated in terms of Dedekind cuts. For basic notions of (recursive) enumerability, we refer to the book of Cutland [Cut80].

Definition 4.1 (right-computable)**.** A real number α is *right-computable* if the set $\{x \in \mathbb{Q} \mid \alpha < x\}$ is recursively enumerable. We say that α is *computable* if both $\{x \in \mathbb{Q} \mid \alpha < x\}$ and $\{x \in \mathbb{Q} \mid \alpha > x\}$ are recursively enumerable.

Further information on different notions of one-sided computability of real numbers can be found in the work of Zheng and Rettinger [ZR04].

There are only countably many recursively enumerable subsets of $\mathbb{Q}$ and thus the set of right computable and computable numbers is countable.

We collect some easy properties:

Lemma 4.2.

1. *If $\alpha, \beta \in \mathbb{R}_{\geq 0}$ are right-computable and non-negative, then so is $\alpha \cdot \beta \in \mathbb{R}$.*
2. *If $\alpha \in \mathbb{R}_{\geq 0}$ is a real number and $c \in \mathbb{R}_{>0}$ a computable number such that $c \cdot \alpha$ is right-computable, then α is right-computable.*

Proof. For the first part we observe that if $\alpha, \beta \geq 0$, then $\{x \in \mathbb{Q} \mid \alpha < x\} \cdot \{y \in \mathbb{Q} \mid \beta < y\} = \{z \in \mathbb{Q} \mid \alpha \cdot \beta < z\}$.

For the second part, let $\alpha \in \mathbb{R}_{\geq 0}$ be such that $c \cdot \alpha$ is right-computable, where c is computable. Since c is computable and positive, so is c^{-1}, thus c^{-1} is in particular right-computable. Hence $\alpha = c^{-1} \cdot (c \cdot \alpha)$ is the product of non-negative right-computable numbers and thus right-computable. □

To a subset $A \in \mathbb{N}$ we associate the number $x_A := \sum_{n \in \mathbb{N}} 2^{-n}$. We relate the (right-)computability of x_A to the computability of A as a subset of $\mathbb{N}$, following Specker [Spe49].

Proposition 4.3. *Let $A \subset \mathbb{N}$ and let x_A be defined as above. Then:*

1. *If the set A is recursively enumerable, then x_A is left-computable and $2 - x_A = x_{\mathbb{N}\setminus A}$ is right-computable.*
2. *The set A is recursive if and only if x_A is computable.*
3. *If A is recursively enumerable but not recursive, then x_A is not right-computable.*

Proof. The first two items are classical results of Specker [Spe49]. To see item 3, let A be recursively enumerable but not recursive. *Assume* that x_A is right-computable. By item 1, x_A is then also left-computable. Thus, x_A is both left- and right-computable, whence computable. But by item 2 this implies that A is recursive, which contradicts our assumption. □

Lemma 4.4. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function with the following property: The set $\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid f(m) \leq n\} \subset \mathbb{N}\times\mathbb{N}$ is recursively enumerable. Then*

$$\inf_{m\in\mathbb{N}_{>0}} \frac{f(m)}{m}$$

is right-computable.

Proof. Set $S := \{(m,n) \in \mathbb{N}\times\mathbb{N} \mid f(m) \leq n\}$ and observe that

$$\inf_{m\in\mathbb{N}_{>0}} \frac{f(m)}{m} = \inf_{(m,n)\in S} \frac{n}{m}.$$

There is a Turing machine that, as input, takes a rational number and then enumerates all rational numbers above it. We may diagonally use this Turing machine and the enumeration of S to enumerate the set

$$\Bigl\{x \in \mathbb{Q} \Bigm| \exists_{(m,n)\in S} \quad \frac{n}{m} < x\Bigr\} = \Bigl\{x \in \mathbb{Q} \Bigm| \inf_{m\in\mathbb{N}_{>0}} \frac{f(m)}{m} < x\Bigr\}.$$

Thus indeed $\inf_{m\in\mathbb{N}_{>0}} \frac{f(m)}{m}$ is right-computable. □

4.2 Proof of Theorem E

Let M be an oriented closed connected manifold and $d := \dim M$. Then M is homotopy equivalent to a finite (simplicial) complex T [Sie68, KS69]; let $f\colon M \to |T|$ be such a homotopy equivalence and for a commutative ring R with unit, let

$$[T]_R := H_d(f;R)\bigl([M]_R\bigr) \in H_d\bigl(|T|;R\bigr).$$

If R is a normed ring, then we write $\|\cdot\|_{1,R}$ for the associated ℓ^1-semi-norm on $H_d(|T|;R)$. Because f is a homotopy equivalence, we have

$$\|M\| = \bigl\|[M]_\mathbb{R}\bigr\|_{1,\mathbb{R}} = \bigl\|[T]_\mathbb{R}\bigr\|_{1,\mathbb{R}}.$$

Moreover, the ℓ^1-semi-norm with $\mathbb{R}$-coefficients can be computed via rational coefficients [Sch05, Lemma 2.9]:

$$\|M\| = \bigl\|[T]_\mathbb{R}\bigr\|_1 = \bigl\|[T]_\mathbb{Q}\bigr\|_{1,\mathbb{Q}} = \inf_{m\in\mathbb{N}_{>0}} \frac{\|m\cdot[T]_\mathbb{Z}\|_{1,\mathbb{Z}}}{m}.$$

The function $m \mapsto \|m\cdot[T]_\mathbb{Z}\|_{1,\mathbb{Z}}$ satisfies the hypothesis of Lemma 4.4 (see Lemma 4.5 below). Applying Lemma 4.4 therefore shows that the number $\|M\|$ is right-computable.

Lemma 4.5. *In this situation, the subset*

$$\bigl\{(m,n) \in \mathbb{N} \times \mathbb{N} \bigm| \|m \cdot [T]_\mathbb{Z}\|_{1,\mathbb{Z}} \leq n\bigr\} \subset \mathbb{N} \times \mathbb{N}$$

is recursively enumerable.

Proof. We can use a straightforward enumeration of combinatorial models of cycles [Löh18, proof of Corollary 5.1]:

First, $H_d(|T|;\mathbb{Z})$ is isomorphic to the simplicial homology $H_d(T;\mathbb{Z})$ of T. Therefore, we can (algorithmically) determine a simplicial cycle z on T that represents the class $[T]_\mathbb{Z}$; this cycle can also be viewed as a singular cycle on $|T|$.

Inductive simplicial approximation of singular simplices shows that for every singular cycle $c \in C_d(|T|;\mathbb{Z})$, there exists a singular cycle $c' \in C_d(|T|;\mathbb{Z})$ with the following properties:

- The cycles c and c' represent the same homology class in $H_d(|T|;\mathbb{Z})$.
- The chain c' is a *combinatorial singular chain*, i.e., all singular simplices in c' are simplicial maps from an iterated barycentric subdivision of Δ^d to an iterated barycentric subdivision of T.

  Here, each singular simplex in c' is the simplicial approximation of a singular simplex in c. In particular, in general, the image of a singular simplex in c' might touch several simplices of T and might pass them several times.
- We have $|c'|_1 \leq |c|_1$.

This allows us to restrict attention to such combinatorial singular chains. Moreover, the following operations can be performed by Turing machines:

- Enumerate all iterated barycentric subdivisions of T and Δ^d.
- Enumerate all simplicial maps between two finite simplicial complexes.
- Hence: Enumerate all combinatorial singular $\mathbb{Z}$-chains of T.
- Check, for given $m \in \mathbb{N}$, whether a combinatorial singular $\mathbb{Z}$-chain on T is a a cycle and represents the class $m \cdot [T]_\mathbb{Z}$ in $H_d(|T|;\mathbb{Z})$ (through comparison with the corresponding iterated barycentric subdivision of z in simplicial homology).
- Compute the 1-norm of a combinatorial singular $\mathbb{Z}$-chain.

In summary, we can enumerate the set $\{(m,c) \mid m \in \mathbb{N}, c \in C(m)\}$, where $C(m)$ is the set of all combinatorial $\mathbb{Z}$-cycles of T that represent $m \cdot [T]_\mathbb{Z}$ in $H_d(|T|;\mathbb{Z})$.

We now consider the following algorithm: Given $m, n \in \mathbb{N}$, we search for elements of 1-norm at most n in $C(m)$.

- If such an element is found (in finitely many steps), then the algorithm terminates and declares that $\|m \cdot [T]_\mathbb{Z}\|_{1,\mathbb{Z}} \leq n$.
- Otherwise the algorithm does not terminate.

From the previous discussion, it is clear that this algorithm witnesses that the set $\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid \|m \cdot [T]_\mathbb{Z}\|_{1,\mathbb{Z}} \leq n\}$ is recursively enumerable. □

This completes the proof of Theorem E.

Remark 4.6. It should be noted that the argument above is constructive enough to also give a slightly stronger statement (similar to the case of integral simplicial volume [Löh18, Remark 5.2]): The function from the set of (finite) simplicial complexes (with vertices in $\mathbb{N}$) that triangulate oriented closed connected manifolds to the set of subsets of $\mathbb{Q}$ given by

$$T \mapsto \| |T| \|$$

is semi-computable (and not only the resulting individual real numbers) in the following sense: There is a Turing machine that given such a triangulation T and $x \in \mathbb{Q}$ as input

- halts if $\| |T| \| < x$ and declares that $\| |T| \| < x$,
- and does not terminate if $\| |T| \| \geq x$.

But it is known that this function is *not* computable [Wei05, Theorem 2, p. 88].

References

[AN98] Alejandro Adem and Nadim Naffah. On the cohomology of $\mathrm{SL}_2(\mathbf{Z}[1/p])$. In *Geometry and cohomology in group theory (Durham, 1994)*, volume 252 of *London Math. Soc. Lecture Note Ser.*, pages 1–9. Cambridge Univ. Press, 1998. Cited on page: 7

[Bak66] A. Baker. Linear forms in the logarithms of algebraic numbers. I, II, III. *Mathematika*, 13:204–216; ibid. 14 (1967), 102–107; ibid. 14 (1967), 220–228, 1966. Cited on page: 5

[Bav91] Christophe Bavard. Longueur stable des commutateurs. *Enseign. Math. (2)*, 37(1-2):109–150, 1991. Cited on page: 11

[Bes40] A. S. Besicovitch. On the linear independence of fractional powers of integers. *J. London Math. Soc.*, 15:3–6, 1940. Cited on page: 5

[BFH16] Michelle Bucher, Roberto Frigerio, and Tobias Hartnick. A note on semi-conjugacy for circle actions. *Enseign. Math.*, 62(3-4):317–360, 2016. Cited on page: 7

[BM02a] Marc Burger and Nicolas Monod. Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.*, 12(2):219–280, 2002. Cited on page: 7

[BM02b] Marc Burger and Nicolas Monod. On and around the bounded cohomology of SL_2. In *Rigidity in dynamics and geometry (Cambridge, 2000)*, pages 19–37. Springer, Berlin, 2002. Cited on page: 7

[BM19] Michelle Bucher and Nicolas Monod. The bounded cohomology of SL_2 over local fields and S-integers. *Int. Math. Res. Not.*, (6):1601–1611, 2019. Cited on page: 7

[Bro94] Kenneth S. Brown. *Cohomology of Groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. Cited on page: 7

[Cal09a] Danny Calegari. *scl*, volume 20 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2009. Cited on page: 2, 3, 7, 11

[Cal09b] Danny Calegari. Stable commutator length is rational in free groups. *J. Amer. Math. Soc.*, 22(4):941–961, 2009. Cited on page: 2

[CF10] Danny Calegari and Koji Fujiwara. Stable commutator length in word-hyperbolic groups. *Groups Geom. Dyn.*, 4(1):59–90, 2010. Cited on page: 2

[CH19] Lvzhou Chen and Nicolaus Heuer. Spectral gap of scl in graphs of groups and 3-manifolds. 2019. arXiv:1910.14146 [math.GT]. Cited on page: 2

[Cut80] Nigel Cutland. *Computability.* Cambridge University Press, Cambridge-New York, 1980. An introduction to recursive function theory. Cited on page: 2, 11

[EHS72] Beno Eckmann, Peter J. Hilton, and Urs Stammbach. On the homology theory of central group extensions. I. The commutator map and stem extensions. *Comment. Math. Helv.*, 47:102–122, 1972. Cited on page: 8

[Eis12] Michael Eisermann. The fundamental theorem of algebra made effective: an elementary real-algebraic proof via Sturm chains. *Amer. Math. Monthly*, 119(9):715–752, 2012. Cited on page: 4

[Ghy87] Étienne Ghys. Groupes d'homéomorphismes du cercle et cohomologie bornée. In *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 81–106. Amer. Math. Soc., Providence, RI, 1987. Cited on page: 7

[Gro82] Michael Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99 (1983), 1982. Cited on page: 1, 10

[GS87] Étienne Ghys and Vlad Sergiescu. Sur un groupe remarquable de difféomorphismes du cercle. *Comment. Math. Helv.*, 62(2):185–239, 1987. Cited on page: 9

[Heu19] Nicolaus Heuer. The full spectrum of scl on recursively presented groups. 2019. arXiv:1909.01309 [math.GR]. Cited on page: 2

[HL19a] Nicolaus Heuer and Clara Löh. Simplicial volume of one-relator groups and stable commutator length. 2019. arXiv:1911.02470 [math.GT]. Cited on page: 2

[HL19b] Nicolaus Heuer and Clara Löh. The spectrum of simplicial volume. 2019. arXiv:1904.04539 [math.GT], to appear in *Invent. math.* Cited on page: 1, 2, 8, 9

[Iva85] N. V. Ivanov. Foundations of the theory of bounded cohomology. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 143:69–109, 177–178, 1985. Studies in topology, V. Cited on page: 10

[KS69] R. C. Kirby and L. C. Siebenmann. On the triangulation of manifolds and the Hauptvermutung. *Bull. Amer. Math. Soc.*, 75:742–749, 1969. Cited on page: 12

[Lim17] Fábio M. S. Lima. Some transcendence results from a harmless irrationality theorem. *J. Ana. Num. Theor.*, 5(2), 2017. Cited on page: 4

[Löh18] Clara Löh. Odd manifolds of small integral simplicial volume. *Ark. Mat.*, 56(2):351–375, 2018. Cited on page: 13, 14

[ML95] Saunders Mac Lane. *Homology.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition. Cited on page: 9

[Mos80] Kenneth N. Moss. Homology of $\mathrm{SL}(n, \mathbf{Z}[1/p])$. *Duke Math. J.*, 47(4):803–818, 1980. Cited on page: 7

[Mun80] Hans J. Munkholm. Simplices of maximal volume in hyperbolic space, Gromov's norm, and Gromov's proof of Mostow's rigidity theorem (following Thurston). In *Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979)*, volume 788 of *Lecture Notes in Math.*, pages 109–124. Springer, Berlin, 1980. Cited on page: 1

[Niv56] Ivan Niven. *Irrational numbers.* The Carus Mathematical Monographs, No. 11. The Mathematical Association of America. Distributed by John Wiley and Sons, 1956. Cited on page: 4

[Sch05] Marco Schmidt. *L^2-Betti Numbers of $\mathcal{R}$-Spaces and the Integral Foliated Simplicial Volume.* PhD thesis, Westfälische Wilhelms-Universität Münster, 2005. http://nbn-resolving.de/urn:nbn:de:hbz:6-05699458563. Cited on page: 12

[Ser03] Jean-Pierre Serre. *Trees.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. Cited on page: 7

[Sie68] L. C. Siebenmann. On the homotopy type of compact topological manifolds. *Bull. Amer. Math. Soc.*, 74:738–742, 1968. Cited on page: 12

[Spe49] Ernst Specker. Nicht konstruktiv beweisbare Sätze der Analysis. *The Journal of Symbolic Logic*, 14(3):145–158, 1949. Cited on page: 11, 12

[Wei05] Shmuel Weinberger. *Computers, rigidity, and moduli.* M. B. Porter Lectures. Princeton University Press, 2005. The large-scale fractal geometry of Riemannian moduli space. Cited on page: 14

[Zhu08] Dongping Zhuang. Irrational stable commutator length in finitely presented groups. *J. Mod. Dyn.*, 2(3):499–507, 2008. Cited on page: 2

[ZR04] Xizhong Zheng and Robert Rettinger. Weak computability and representation of reals. *MLQ Math. Log. Q.*, 50(4-5):431–442, 2004. Cited on page: 11

Nicolaus Heuer

DPMMS, University of Cambridge
nh441@cam.ac.uk, https://www.dpmms.cam.ac.uk/~nh441

Clara Löh

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg
clara.loeh@mathematik.uni-r.de, http://www.mathematik.uni-r.de/loeh