SIMPLICIAL VOLUME WITH $\mathbb{F}_p$-COEFFICIENTS

CLARA LÖH

ABSTRACT. For primes p, we investigate an $\mathbb{F}_p$-version of simplicial volume and compare these invariants with their siblings over other coefficient rings. We will also consider the associated gradient invariants, obtained by stabilisation along finite coverings. Throughout, we will discuss the relation between such simplicial volumes and Betti numbers.

1. INTRODUCTION

Simplicial volumes measure the size of manifolds in terms of singular fundamental cycles. Different choices of coefficients and of counting simplices in singular cycles lead to different versions of simplicial volume.

The classical simplicial volume $\| \cdot \|_{\mathbb{R}}$ of an oriented closed connected manifold is defined in terms of the ℓ^1-norm on the singular chain complex with real coefficients, i.e., we use a weighted count of singular simplices [5, 8] (Section 2.1 contains a precise definition).

We will study *weightless* simplicial volumes, i.e., we will ignore the weight of the coefficients and only count the number of singular simplices. In this way, for *every* coefficient ring R, we obtain a notion of weightless simplicial volume $\| \cdot \|_{(R)}$ (Definition 2.1). In particular, this leads to an $\mathbb{F}_p$-version of simplicial volume:

Definition 1.1. Let M be an oriented closed connected n-manifold and let $p \in \mathbb{N}$ be a prime number. Then the $\mathbb{F}_p$-*simplicial volume* $\|M\|_{(\mathbb{F}_p)}$ *of* M is defined as

$$\min\left\{ m \in \mathbb{N} \ \middle| \ \sum_{j=1}^m a_j \cdot \sigma_j \in C_n(M; \mathbb{F}_p) \text{ is an } \mathbb{F}_p\text{-fundamental cycle of } M \right\}.$$

The following natural questions emerge:
  (A) What is the topological/geometric meaning of these invariants?
  (B) How do these invariants relate to other topological invariants?
  (C) (How) Does the choice of the prime number affect simplicial volume?
  (D) Do these invariants relate to the weightless $\mathbb{Q}$-simplicial volume?
  (E) What happens if $\mathbb{F}_p$-simplicial volumes are stabilised along a tower of finite coverings?

In the present paper, we will study these questions, keeping a focus on the comparison with Betti numbers. In order to simplify the discussion, we will start with Questions (C) and (D).

1.1. **Comparison of coefficients.** Using different finite fields as coefficients for singular homology, in general, leads to different Betti numbers. But the universal coefficient theorem implies that this dependence on the coefficients is rather limited: If X is a finite CW-complex and $k \in \mathbb{N}$, then for all but finitely many primes $p \in \mathbb{N}$ we have

$$b_k(X; \mathbb{F}_p) = b_k(X; \mathbb{Q}).$$

The situation for weightless simplicial volumes is similar: In general, different prime numbers will lead to different simplicial volumes over the corresponding finite fields (Example 2.7). But for every manifold, this exceptional behaviour is limited to a finite number of primes:

Theorem 1.2. *Let M be an oriented closed connected manifold. Then for all but finitely many primes $p \in \mathbb{N}$ we have*

$$\|M\|_{(\mathbb{F}_p)} = \|M\|_{(\mathbb{Q})}.$$

In the same spirit, as for Betti numbers, the weightless simplicial volumes of fields with equal characteristic coincide (Theorem 3.8).

In contrast with the Betti number case, the weightless $\mathbb{Q}$-simplicial volume does *not* necessarily coincide with the weightless $\mathbb{Z}$-simplicial volume (Proposition 2.8 shows that $\mathbb{R}P^3$ is an example of this type) and that the ordinary integral simplicial volume $\| \cdot \|_{\mathbb{Z}}$ (Section 2.1) tends to be much bigger than the weightless $\mathbb{Q}$-simplicial volume:

Theorem 1.3. *For every $K \in \mathbb{N}$ there exists an oriented closed connected 3-manifold with*

$$\|M\|_{\mathbb{Z}} > K \cdot \|M\|_{(\mathbb{Q})}.$$

Question 1.4 (weight problem)**.** Let M be an oriented closed connected manifold. What is the difference between integral simplicial volume $\|M\|_{\mathbb{Z}}$ and the weightless integral simplicial volume $\|M\|_{(\mathbb{Z})}$?

1.2. **Topological meaning.** We will now come to questions (A) and (B). Weightless simplicial volumes give Betti number bounds and provide obstructions against domination of manifolds (Definition 2.4). We explain this in more detail:

If M is an oriented closed connected manifold and $p \in \mathbb{N}$ is a prime number, then Poincaré duality shows that for all $k \in \mathbb{N}$ we have

$$b_k(M; \mathbb{F}_p) \leq \|M\|_{(\mathbb{F}_p)}$$

(Proposition 2.6). This estimate can be used to calculate weightless simplicial volumes in simple cases (Section 2.3); conversely, estimates of this type lead to homology gradient bounds, when stabilising along towers of finite coverings (Section 1.3).

Classical simplicial volume is a homotopy invariant and compatible with mapping degrees: If $f \colon M \longrightarrow N$ is a continuous map between oriented closed connected manifolds of the same dimension, then

$$|\deg f| \cdot \|N\|_{\mathbb{R}} \leq \|M\|_{\mathbb{R}}.$$

Therefore, classical simplicial volume gives a priori bounds on the set of possible mapping degrees [5].

Similarly, also weightless simplicial volumes are homotopy invariants and they satisfy the following estimates (Proposition 2.3): Let $f\colon M \longrightarrow N$ be a continuous map between oriented closed connected manifolds of the same dimension.

- If $p \in \mathbb{N}$ is a prime number with $p \nmid \deg f$, then $\|N\|_{(\mathbb{F}_p)} \leq \|M\|_{(\mathbb{F}_p)}$.
- If $\deg f \neq 0$, then $\|N\|_{(\mathbb{Q})} \leq \|M\|_{(\mathbb{Q})}$.

In addition to the similarity with the degree estimate for classical simplicial volume these monotonicity properties are similar to the corresponding monotonicity statements for Betti numbers with $\mathbb{F}_p$-coefficients and $\mathbb{Q}$-coefficients, respectively. In particular, the weightless $\mathbb{Q}$-simplicial volume complements the Betti number obstruction against domination of manifolds (Section 2.2).

1.3. **Stabilisation along finite coverings.** Finally, we address the last question (E) on stabilisation of $\mathbb{F}_p$-simplicial volumes along towers of finite coverings.

Definition 1.5. Let M be an oriented closed connected manifold and let $p \in \mathbb{N}$ be a prime number. Then the *stable $\mathbb{F}_p$-simplicial volume of M* is defined as

$$\|M\|_{(\mathbb{F}_p)}^{\infty} := \inf \left\{ \frac{\|N\|_{(\mathbb{F}_p)}}{d} \; \Big| \; d \in \mathbb{N}, \, N \to M \text{ is a } d\text{-sheeted covering} \right\} \in \mathbb{R}_{\geq 0}.$$

Remark 1.6 (inherited vanishing). Let M be an oriented closed connected manifold and let $p \in \mathbb{N}$ be a prime number. Then reduction modulo p shows that

$$\|M\|_{(\mathbb{F}_p)}^{\infty} \leq \|M\|_{\mathbb{Z}}^{\infty}.$$

In particular, all known vanishing results for the stable integral simplicial volume $\|M\|_{\mathbb{Z}}^{\infty}$ imply corresponding vanishing results for $\|M\|_{(\mathbb{F}_p)}^{\infty}$. This includes, for example, the case of aspherical manifolds with residually finite amenable fundamental group [4], smooth aspherical manifolds with non-trivial S^1-action [2], and graph manifolds [3]. Moreover, also aspherical manifolds with small enough amenable covers have vanishing stable weightless simplicial volume [14] (Example 4.12).

In addition, we have the following vanishing phenomenon in the presence of non-trivial self-maps: If M is an oriented closed connected manifold that admits a self-map of non-trivial degree, then the classical simplicial volume satisfies $\|M\|_{\mathbb{R}} = 0$ (this follows from the degree estimate). If M is aspherical and the fundamental group is residually finite, then this vanishing also carries over to L^2-Betti numbers [11, Theorem 14.40]. In the same way, we obtain vanishing of stable weightless simplicial volumes:

Theorem 1.7. *Let M be an oriented closed connected aspherical manifold with residually finite fundamental group. If M admits a continuous map $f\colon M \longrightarrow M$ with $\deg f \notin \{-1, 0, 1\}$, then for all primes $p \in \mathbb{N}$ with $p \nmid \deg f$ we have*

$$\|M\|_{(\mathbb{F}_p)}^{\infty} = 0 = \|M\|_{(\mathbb{Q})}^{\infty}.$$

It should be noted that it is an open problem to determine whether the same conclusion also holds for $\|M\|_{\mathbb{Z}}^{\infty}$ instead of $\|M\|_{(\mathbb{F}_p)}^{\infty}$.

As in the case of stable integral simplicial volume [10, Theorem 6.6, Remark 6.7], stable $\mathbb{F}_p$-simplicial volumes admit a description in terms of the probability measure preserving action of $\pi_1(M)$ on its profinite completion $\widehat{\pi_1(M)}$ (see Section 4.4 for the definitions):

Theorem 1.8 (dynamical view). *Let M be an oriented closed connected manifold with residually finite fundamental group and let $p \in \mathbb{N}$ be a prime number. Then*

$$\|M\|_{(\mathbb{F}_p)}^\infty = \|M\|_{(\widehat{\pi_1(M)};\mathbb{F}_p)}.$$

We conclude with a brief outlook on the relation with homology gradients:

Remark 1.9 (homology gradients). Let M be an oriented closed connected manifold and let $p \in \mathbb{N}$ be a prime number. Let $(\Gamma_j)_{j \in \mathbb{N}}$ be a descending chain of finite index subgroups of $\pi_1(M)$ and let $(M_j)_{j \in \mathbb{N}}$ be the corresponding tower of covering manifolds. Then the Betti number bound of Proposition 2.6 yields the corresponding homology gradient bound

$$\limsup_{j \to \infty} \frac{b_k(M_j;\mathbb{F}_p)}{[\pi_1(M):\Gamma_j]} \leq \lim_{j \to \infty} \frac{\|M_j\|_{(\mathbb{F}_p)}}{[\pi_1(M):\Gamma_j]} = \inf_{j \in \mathbb{N}} \frac{\|M_j\|_{(\mathbb{F}_p)}}{[\pi_1(M):\Gamma_j]}.$$

Similar to the case of homology gradients, we are therefore led to the following open problems:

Question 1.10 (approximation problem). Let M be an oriented closed connected manifold with residually finite fundamental group. For which (if any) primes $p \in \mathbb{N}$ do we have

$$\|M\|_{(\mathbb{F}_p)}^\infty = \|M\|_{(\mathbb{Q})}^\infty \ ?$$

Question 1.11 (chain problem). Let M be an oriented closed connected manifold with residually finite fundamental group, let $p \in \mathbb{N}$ be a prime, let $(\Gamma_j)_{j \in \mathbb{N}}$ be a descending chain of finite index subgroups of $\pi_1(M)$, and let $(M_j)_{j \in \mathbb{N}}$ be the associated sequence of covering manifolds of M. How does the value

$$\lim_{j \to \infty} \frac{\|M_j\|_{(\mathbb{F}_p)}}{[\pi_1(M):\Gamma_j]}$$

depend on the choice of this chain $(\Gamma_j)_{j \in \mathbb{N}}$?

Question 1.12 (Euler characteristic problem). Gromov asked the following question [6, p. 232]: Let M be an oriented closed connected aspherical manifold M with $\|M\|_\mathbb{R} = 0$. Does this imply $\chi(M) = 0$?

The Betti number estimate shows that $|\chi(M)| \leq (\dim M + 1) \cdot \|M\|_{(\mathbb{F}_p)}^\infty$ for all primes p (Proposition 4.4).

Hence, we arrive at the following question: Let M be an oriented closed connected aspherical manifold with $\|M\|_\mathbb{R} = 0$ (and residually finite fundamental group). Does there exist a prime number $p \in \mathbb{N}$ with $\|M\|_{(\mathbb{F}_p)}^\infty = 0$?

Organisation of this article. We introduce weightless simplicial volumes in Section 2 and establish some basic properties, including a proof of Theorem 1.3. In Section 3 we prove the comparison theorem Theorem 1.2. Stable weightless simplicial volumes are studied in Section 4, where we prove Theorem 1.7 and Theorem 1.8.

2. Weightless simplicial volumes

We will introduce weightless simplicial volumes and the special case of $\mathbb{F}_p$-simplicial volumes. Moreover, we will collect some basic properties of these invariants.

2.1. **Basic definitions.** If R is a subring of $\mathbb{R}$, then the *R-simplicial volume* of an oriented closed connected n-manifold M is defined as

$$\|M\|_R := \inf\left\{\sum_{j=1}^{m} |a_j| \;\Big|\; \sum_{j=1}^{m} a_j \cdot \sigma_j \in C_n(M; R)\right.$$

$$\left. \text{is an } R\text{-fundamental cycle of } M\right\} \in \mathbb{R}_{\geq 0}.$$

Here, $|\cdot|$ denotes the standard absolute value on $\mathbb{R}$ and $C_n(M; R)$ is the n-th chain group of the singular chain complex of M. Classical simplicial volume, as introduced by Gromov [5], is $\|M\|_{\mathbb{R}}$.

If R is a commutative ring with unit, in general, we will not find a norm on R in the usual sense; but we can always use the trivial size function

$$|\cdot| \colon R \longrightarrow \mathbb{R}_{\geq 0}$$

$$x \longmapsto \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

on the coefficients, which leads to the counting "norm" on the singular chain complex (thereby forgetting the weights of the individual simplices): If X is a topological space and $k \in \mathbb{N}$, then we define

$$\left|\sum_{j=1}^{m} a_j \cdot \sigma_j\right| := \sum_{j=1}^{m} |a_j| = m$$

for all singular chains $\sum_{j=1}^{m} a_j \cdot \sigma_j \in C_k(X; R)$ in reduced form. This "norm" on the singular chain complex results in the following definition:

Definition 2.1 (weightless simplicial volume). Let R be a commutative ring with unit and let M be an oriented closed connected manifold of dimension n. Then the *weightless R-simplicial volume* of M is defined as

$$\|M\|_{(R)} := \min\left\{m \in \mathbb{N} \;\Big|\; \sum_{j=1}^{m} a_j \cdot \sigma_j \in C_n(M; R)\right.$$

$$\left. \text{is an } R\text{-fundamental cycle of } M\right\} \in \mathbb{N}.$$

If $p \in \mathbb{N}$ is a prime number, then for the ring $\mathbb{F}_p$ we obtain Definition 1.1.

Of course, we can extend this definition of $\|\cdot\|_{(R)}$ to all singular homology classes with R-coefficients by minimising the counting norm over all singular cycles representing the given homology class.

Remark 2.2. Let R be a (non-trivial) commutative ring with unit and let M be an oriented closed connected manifold.

(1) If M is non-empty, then $\|M\|_{(R)} \neq 0$, whence $\|M\|_{(R)} \geq 1$.

(2) Because every $\mathbb{Z}$-fundamental cycle gives rise to an R-fundamental cycle, we have

$$\|M\|_{(R)} \leq \|M\|_{(\mathbb{Z})} \leq \|M\|_{\mathbb{Z}}.$$

2.2. **Degrees and domination.** Weightless simplicial volumes can serve as an obstruction against domination of manifolds.

Proposition 2.3 (degree monotonicity). *Let M and N be oriented closed connected manifolds of the same dimension and let $f \colon M \longrightarrow N$ be a continuous map.*

(1) *If R is a commutative ring with unit and $\deg f$ is a unit in R, then*

$$\|N\|_{(R)} \leq \|M\|_{(R)}.$$

(2) *In particular: If $\deg f \neq 0$, then $\|N\|_{(\mathbb{Q})} \leq \|M\|_{(\mathbb{Q})}$. If $p \in \mathbb{N}$ is prime and $p \nmid \deg f$, then $\|N\|_{(\mathbb{F}_p)} \leq \|M\|_{(\mathbb{F}_p)}$.*

Proof. Let $c = \sum_{j=1}^{k} a_j \cdot \sigma_j \in C_n(M; R)$ be an R-fundamental cycle that is in reduced form. Because $\deg f$ is a unit in R, the chain

$$c' := \sum_{j=1}^{k} \frac{1}{\deg f} \cdot a_j \cdot f \circ \sigma_j \in C_n(N; R)$$

is an R-fundamental cycle of N. In particular,

$$\|N\|_{(R)} \leq |c'| \leq k = |c|.$$

Taking the minimum over all R-fundamental cycles c of M shows that $\|N\|_{(R)} \leq \|M\|_{(R)}$.

The second part is a special case of the first part. $\qquad\square$

Definition 2.4 (domination). Let M and N be oriented closed connected manifolds of the same dimension. The manifold M *dominates* N if there exists a map $M \longrightarrow N$ of non-zero degree.

Corollary 2.5. *Let M and N be oriented closed connected manifolds of the same dimension and let F be a field of characteristic 0. If M dominates N, then*

$$\|M\|_{(F)} \geq \|N\|_{(F)}.$$

Proof. This is merely a reformulation of Proposition 2.3, using the domination terminology. $\qquad\square$

2.3. **Homology bounds and some examples.** We will now explain how homology bounds in terms of weightless simplicial volumes can be used to compute first examples.

Proposition 2.6 (homology bounds). *Let M be an oriented closed connected manifold and let R be a commutative ring with unit.*

(1) *For all $\alpha \in H_*(M; R)$ we have $\|\alpha\|_{(R)} \leq \|M\|_{(R)}$.*
(2) *If R is a principal ideal domain and $k \in \mathbb{N}$, then*

$$\mathrm{rk}_R H_k(M; R) \leq \|M\|_{(R)}.$$

Proof. Like all results of this type, this is based on exploiting the explicit description of the Poincaré duality map.

Let $n := \dim M$, let $c = \sum_{j=1}^{m} a_j \cdot \sigma_j \in C_n(M; R)$ be an R-fundamental cycle of M with $m = \|M\|_{(R)}$, and let $k \in \mathbb{N}$. Then the Poincaré duality map

$$\cdot \cap [M]_R \colon H^{n-k}(M; R) \longrightarrow H_k(M; R)$$

$$[f] \longmapsto (-1)^{(n-k)\cdot k} \cdot \left[\sum_{j=1}^{m} a_j \cdot f(_{n-k}\lfloor \sigma_j \rfloor) \cdot \sigma_j \rfloor_k \right]$$

is an isomorphism of R-modules. Clearly, the elements on the right hand side have weightless norm at most $m = |c| = \|M\|_{(R)}$. This proves the first part.

Moreover, the R-module $H_k(M; R)$ is a quotient of a submodule of an R-module that is generated by m elements. If R is a principal ideal domain, this implies $\mathrm{rk}_R H_k(M; R) \le m$. $\qquad\square$

Example 2.7 (odd-dimensional projective spaces)**.** Let $n \in \mathbb{N}$ be odd. If $p \in \mathbb{N}$ is an odd prime, then

$$\|\mathbb{R}P^n\|_{(\mathbb{F}_p)} = 1$$

because $\mathbb{R}P^n$ is dominated by S^n through a map of degree 2 and 2 is invertible in $\mathbb{F}_p$ (Proposition 2.3).

In contrast,

$$\|\mathbb{R}P^n\|_{(\mathbb{F}_2)} = 2.$$

Indeed, on the one hand, $\|\mathbb{R}P^n\|_{(\mathbb{F}_2)} \le \|\mathbb{R}P^n\|_{\mathbb{Z}} = 2$ [9, Proposition 4.4] (the cited argument also generalises to odd dimensions bigger than 3). On the other hand, let us assume for a contradiction that $\|\mathbb{R}P^n\|_{(\mathbb{F}_2)} = 1$. Then the Poincaré duality argument from Proposition 2.6 shows that the non-trivial class in $H_2(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2$ can be represented by a single singular 2-simplex; however, a single singular 2-simplex cannot be an $\mathbb{F}_2$-cycle (for parity reasons). This contradiction shows that $\|\mathbb{R}P^n\|_{(\mathbb{F}_2)} = 2$.

Proposition 2.8 (minimal weightless simplicial volume)**.** *Let M be an oriented closed connected manifold and let $n := \dim M$.*

(1) *Then $\|M\|_{(\mathbb{Q})} = 1$ if and only if n is odd and M is dominated by S^n.*
(2) *Then $\|M\|_{(\mathbb{Z})} = 1$ if and only if n is odd and $M \simeq S^n$.*

Proof. If n is odd, then $\|S^n\|_{\mathbb{Z}} = 1$ (we can wrap Δ^n around S^n with constant face maps). Hence: If $M \simeq S^n$, then $\|M\|_{(\mathbb{Z})} = 1$. If M is dominated by S^n, then $1 \le \|M\|_{(\mathbb{Q})} \le \|S^n\|_{(\mathbb{Q})} = 1$.

Conversely, if $\|M\|_{(\mathbb{Q})} = 1$, then there is a singular simplex $\sigma \colon \Delta^n \longrightarrow M$ that is a cycle that represents a non-zero homology class in $H_n(M; \mathbb{Q})$ and in $H_n(M; \mathbb{Z})$. In particular, there is an $m \in \mathbb{Z} \setminus \{0\}$ with $\|m \cdot [M]_{\mathbb{Z}}\|_{\mathbb{Z}} = 1$. It is known that this implies that n is odd and that M is dominated by S^n [9, Theorem 3.2].

Similarly, if $\|M\|_{(\mathbb{Z})} = 1$, then there is a singular simplex $\sigma \colon \Delta^n \longrightarrow M$ and an $m \in \mathbb{Z} \setminus \{0\}$ such that $m \cdot \sigma$ is a fundamental cycle of M. Because $[m \cdot \sigma] = [M]_{\mathbb{Z}}$ is a generator of $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$, we obtain $m \in \{-1, 1\}$.

Therefore, $\|M\|_{\mathbb{Z}} = 1$, which implies that n is odd and $M \simeq S^n$ [9, Theorem 1.1]. $\qquad\square$

We will now establish Theorem 1.3, by proving the following generalisation:

Theorem 2.9. *Let F be a field. For every $K \in \mathbb{N}$ there exists an oriented closed connected 3-manifold with*

$$\|M\|_{\mathbb{Z}} > K \cdot \|M\|_{(F)}.$$

Proof. Let p be the characteristic of F and let $N := \mathbb{N} \setminus p \cdot \mathbb{N}$. For $n \in \mathbb{N}$, we write $L(n, 1)$ for the associated 3-dimensional lens space. Then $L(n, 1)$ is covered, whence dominated, by S^3 through a map of degree n; therefore, for all $n \in N$ we obtain

$$\big\|L(n, 1)\big\|_{(F)} \leq \|S^3\|_{(F)} \leq \|S^3\|_{\mathbb{Z}} = 1$$

(by Proposition 2.3).

On the other hand, the manifolds in the sequence $\big(L(n, 1)\big)_{n \in N}$ are pairwise non-homeomorphic. Therefore, the sequence $\big(\|L(n, 1)\|_{\mathbb{Z}}\big)_{n \in N}$ is unbounded by the finiteness result for the integral simplicial volume $\|\cdot\|_{\mathbb{Z}}$ on 3-manifolds [9, Proposition 5.3]. $\qquad\square$

Classical simplicial volume (with real coefficients) is known to behave well with respect to products [5]. For integral simplicial volume of a product manifold, there is an upper bound in terms of the products of the integral simplicial volumes of the factors; however, it is unknown whether the integral simplicial volume also satisfies a corresponding estimate from below.

Proposition 2.10 (product estimate). *Let M and N be oriented closed connected manifolds and let R be a commutative ring with unit. Then*

$$\max\big(\|M\|_{(R)}, \|N\|_{(R)}\big) \leq \|M \times N\|_{(R)} \leq \binom{\dim M + \dim N}{\dim M} \cdot \|M\|_{(R)} \cdot \|N\|_{(R)}.$$

Proof. The upper estimate can be shown via the explicit description of the homological cross product on the level of singular chains through the shuffle product; this is similar to the case of ordinary simplicial volume [1, Theorem F.2.5].

For the lower estimate, we argue as follows: Let $y \in N$ be a point and let $i \colon M \longrightarrow M \times \{y\} \longrightarrow M \times N$ and $p \colon M \times N \longrightarrow M$ be the corresponding inclusion and projection, respectively; moreover, we consider

$$\alpha := H_{\dim M}(i; R)[M]_R \in H_{\dim M}(M \times N; R).$$

Using Proposition 2.6, we obtain

$$\|M\|_{(R)} = \big\|H_{\dim M}(p; R)(\alpha)\big\|_{(R)} \leq \|\alpha\|_{(R)} \leq \|M \times N\|_{(R)}.$$

In the same way, we also have $\|N\|_{(R)} \leq \|M \times N\|_{(R)}$. $\qquad\square$

Using the Betti number estimate, we can also give an answer to the weight problem (Question 1.4) in a very simple case:

Proposition 2.11. *Let M be an oriented closed connected manifold and let $k \in \mathbb{N}$. Then $b_k(M; \mathbb{Z}) = \|M\|_{\mathbb{Z}}$ if and only if $b_k(M; \mathbb{Z}) = \|M\|_{(\mathbb{Z})}$.*

Proof. The Betti number estimate (Proposition 2.6) and the universal coefficient theorem imply that

$$b_k(M;\mathbb{Z}) \leq b_k(M;\mathbb{F}_p) \leq \|M\|_{(\mathbb{F}_p)} \leq \|M\|_{(\mathbb{Z})} \leq \|M\|_{\mathbb{Z}}.$$

for all primes $p \in \mathbb{N}$. Hence, if $b_k(M;\mathbb{Z}) = \|M\|_{\mathbb{Z}}$, then $b_k(M;\mathbb{Z}) = \|M\|_{(\mathbb{Z})}$ (and also $b_k(M;\mathbb{Z}) = \|M\|_{(\mathbb{F}_p)}$ for all primes p).

Conversely, let us suppose that $b_k(M;\mathbb{Z}) = \|M\|_{(\mathbb{Z})}$. Let $n := \dim M$ and let $c = \sum_{j=1}^{m} a_j \cdot \sigma_j \in C_n(M;\mathbb{Z})$ be a $\mathbb{Z}$-fundamental cycle of M that satisfies $m = \|M\|_{(\mathbb{Z})}$. *Assume* for a contradiction that there exists a $j \in \{1,\ldots,m\}$ with $a_j \notin \{-1,1\}$. Then there is a prime number $p \in \mathbb{N}$ that divides a_j. Hence, the term $a_j \cdot \sigma_j$ vanishes after reduction modulo p and so

$$b_k(M;\mathbb{Z}) \leq \|M\|_{(\mathbb{F}_p)} \leq m - 1 < m = \|M\|_{(\mathbb{Z})} = b_k(M;\mathbb{Z}),$$

which is a contradiction. This shows that $a_1,\ldots,a_m \in \{-1,1\}$. Therefore, we have

$$\|M\|_{\mathbb{Z}} \leq |c|_1 = m = \|M\|_{(\mathbb{Z})} \leq \|M\|_{\mathbb{Z}},$$

and thus $\|M\|_{\mathbb{Z}} = \|M\|_{(\mathbb{Z})}$. $\square$

Example 2.12 (surfaces). Let Σ be an oriented closed connected (non-empty) surface and let $p \in \mathbb{N}$ be a prime.

If $\Sigma \cong S^2$, then $\|\Sigma\|_{(\mathbb{F}_p)} \geq 2$ (for parity reasons) and $\|\Sigma\|_{(\mathbb{F}_p)} \leq 2$ (straightforward construction). Hence, $\|S^2\|_{(\mathbb{F}_p)} = 2$. Similarly, $\|S^2\|_{(R)} = 2$ for every non-trivial commutative ring R with unit.

We will now consider the case $\Sigma \not\cong S^2$. Let $g \in \mathbb{N}$ denote the genus of Σ. Because $\|\Sigma\|_{\mathbb{Z}} = 4 \cdot g - 2$ [9, Proposition 4.3], we obtain the upper estimate

$$\|\Sigma\|_{(\mathbb{F}_p)} \leq \|\Sigma\|_{(\mathbb{Z})} \leq \|\Sigma\|_{\mathbb{Z}} = 4 \cdot g - 2.$$

On the other hand, the Betti number estimate (Proposition 2.6) yields

$$2 \cdot g = b_1(\Sigma;\mathbb{F}_p) \leq \|\Sigma\|_{(\mathbb{F}_p)} \quad \text{and} \quad 2 \cdot g = b_1(\Sigma;\mathbb{Z}) \leq \|\Sigma\|_{(\mathbb{Z})}.$$

In particular, $\|S^1 \times S^1\|_{(\mathbb{F}_p)} = 2 = \|S^1 \times S^1\|_{(\mathbb{Z})}$. However, if $g \geq 2$, the exact values of $\|\Sigma\|_{(\mathbb{F}_p)}$ or $\|\Sigma\|_{(\mathbb{Z})}$ are not known.

For the weightless simplicial volume with $\mathbb{Z}$-coefficients, we can at least improve the bound $2 \cdot g \leq \|\Sigma\|_{(\mathbb{Z})}$ to a strict inequality: *Assume* for a contradiction that $2 \cdot g = \|\Sigma\|_{(\mathbb{Z})}$. Then Proposition 2.11 implies that $2 \cdot g = \|\Sigma\|_{\mathbb{Z}} = 4 \cdot g - 2$, which is impossible for $g \geq 2$. Hence, we obtain $2 \cdot g < \|\Sigma\|_{(\mathbb{Z})}$.

3. Comparison theorems

We will now focus on the comparison theorem (Theorem 1.2) that relates weightless simplicial volumes over $\mathbb{F}_p$ and $\mathbb{Q}$. In order to promote fundamental cycles over $\mathbb{F}_p$ to fundamental cycles over $\mathbb{Q}$ (while keeping control on the number of simplices), we will consider the combinatorial types of simplices and encode the relevant information in (finite) systems of linear equations.

3.1. **Combinatorics of singular cycles.** We record the combinatorial structure of singular chains in certain generalised simplicial complexes. These complexes are specified by a set of n-simplices and an adjacency relation between the faces of these n-simplices.

Definition 3.1 (model complexes). Let $n \in \mathbb{N}$. An n-*dimensional model complex* is a pair $Z = (S, \sim)$ consisting of a set S (the set of *simplices of Z*) and an equivalence relation $\sim$ on $S \times \{0, \ldots, n\}$.

Two n-dimensional model complexes $(S, \sim)$ and $(S', \sim')$ are *isomorphic* if there exists a bijection $f \colon S \longrightarrow S'$ between their sets of simplices that is compatible with the adjacency relations, i.e.,

$$\forall_{(s,j),(t,k) \in S \times \{0,\ldots,n\}} \quad (s,j) \sim (t,k) \Longleftrightarrow \big(f(s),j\big) \sim' \big(f(t),k\big).$$

Definition 3.2 (model). Let M be a topological space, let $n \in \mathbb{N}$, let R be a commutative ring with unit, and let $c = \sum_{j=1}^m a_j \cdot \sigma_j \in C_n(M; R)$ be a singular chain in reduced form. The *model of c* is the n-dimensional model complex $Z = (\{\sigma_1, \ldots, \sigma_m\}, \sim)$, where $\sim$ is given by

$$\forall_{\sigma,\tau \in \{\sigma_1,\ldots,\sigma_m\}} \quad \forall_{j,k \in \{0,\ldots,n\}} \quad (\sigma,j) \sim (\tau,k) \Longleftrightarrow \sigma \circ \partial_j = \tau \circ \partial_k.$$

Here $\partial_j \colon \Delta^{n-1} \longrightarrow \Delta^n$ denotes the inclusion of the j-th face of Δ^n.

3.2. **Translation to linear algebra.** Because the model of a singular chain stores which faces are equal, we can encode the property of being a cycle into a linear equation. (The linear equation is redundant, but it does have the advantage that the description is simple.)

Definition 3.3 (cycle matrix). Let $n \in \mathbb{N}$ and let $Z = (S, \sim)$ be an n-dimensional model complex. If $(s,i), (t,j) \in S$, then we write

$$r_{(s,i),(t,j)} := \begin{cases} 0 & \text{if } (s,i) \nsim (t,j) \\ (-1)^j & \text{if } (s,i) \sim (t,j). \end{cases}$$

The *cycle matrix of Z* is the matrix $A = (a_{(s,i),t})_{((s,i),t) \in (S \times \{0,\ldots,n\}) \times S}$ given by

$$a_{(s,i),t} := \sum_{j=0}^n r_{(s,i),(t,j)}$$

for all $(s,i) \in S \times \{0, \ldots, n\}$, $t \in S$.

Lemma 3.4. *Let M be a topological space, let $n \in \mathbb{N}$, let R be a commutative ring with unit. Let $c = \sum_{j=1}^m a_j \cdot \sigma_j \in C_n(M; R)$ be a singular chain in reduced form, and let A be the cycle matrix of the model of c. Then c is a cycle if and only if*

$$A \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = 0.$$

Here, we view A as a matrix over R via the canonical unital ring homomorphism $\mathbb{Z} \longrightarrow R$.

Proof. Let $(s,i) \in S \times \{0, \ldots, n\}$ and let $A_{(s,i)}$ be the (s,i)-row of A. By construction of the cycle matrix A, the value

$$A_{(s,i)} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \quad \text{in } R$$

is the total contribution of the singular simplex $s \circ \partial_i$ in the chain $\partial c = \sum_{j=1}^m \sum_{\ell=0}^n (-1)^\ell \cdot a_j \cdot \sigma_j \circ \partial_\ell \in C_{n-1}(M; R)$.

Hence, $A \cdot (a_1, \ldots, a_m)^\top = 0$ if and only if $\partial c = 0$. $\square$

3.3. **Basics on solution spaces.** For the sake of completeness, we recall two facts on solutions of linear equations:

Definition 3.5. Let $k, m \in \mathbb{N}$, let $A \in M_{k \times m}(\mathbb{Z})$ and $b \in \mathbb{Z}^k$. If R is a commutative ring with unit, then we view A and b as a matrix/vector over R and we write

$$L_A(R) := \{x \in R^m \mid A \cdot x = 0\} \subset R^m$$
$$L_{A,b}(R) := \{x \in R^m \mid A \cdot x = b\} \subset R^m$$

for the corresponding sets of solutions over R.

Lemma 3.6. *Let F and E be fields of the same characteristic. Let $k, m \in \mathbb{N}$ and let $A \in M_{k \times m}(\mathbb{Z})$, $b \in \mathbb{Z}^k$. Then*

$$L_{A,b}(F) \neq \emptyset \Longleftrightarrow L_{A,b}(E) \neq \emptyset.$$

Proof. By transitivity, it suffices to consider the case where F is the prime field of E. In view of Gaussian elimination, there exist invertible matrices $S \in \mathrm{GL}_k(F)$ and $T \in \mathrm{GL}_m(F)$ such that

$$\widetilde{A} := S \cdot A \cdot T \in M_{k \times m}(F)$$

is in row echelon form (because the original equation is defined over $\mathbb{Z}$, whence over F, we only need matrices S and T with entries in F).

Let $\widetilde{b} := S \cdot b \cdot T \in F^k$. Then

$$L_{A,b}(F) \neq \emptyset \Longleftrightarrow L_{\widetilde{A},\widetilde{b}}(F) \neq \emptyset$$

and similarly for E. Moreover, because $\widetilde{A}$ is in row echelon form and F is a subfield of E, we have

$$L_{\widetilde{A},\widetilde{b}}(F) \neq \emptyset \Longleftrightarrow L_{\widetilde{A},\widetilde{b}}(E) \neq \emptyset.$$

Combining these equivalences proves the claim. $\square$

Lemma 3.7. *Let $k, m \in \mathbb{N}$ and let $A \in M_{k \times m}(\mathbb{Z})$. Suppose that $P \subset \mathbb{N}$ is an infinite set of primes with*

$$\forall_{p \in P} \quad L_A(\mathbb{F}_p) \neq \{0\}.$$

Then $L_A(\mathbb{Z}) \neq \{0\}$ and for all but finitely many $p \in P$ we have (where $\pi_p \colon \mathbb{Z}^m \longrightarrow \mathbb{F}_p^m$ denotes the reduction modulo p)

$$\pi_p\big(L_A(\mathbb{Z})\big) = L_A(\mathbb{F}_p).$$

Proof. The structure theory of matrices over $\mathbb{Z}$ shows that there exist invertible integral matrices $S \in \mathrm{GL}_k(\mathbb{Z})$ and $T \in \mathrm{GL}_m(\mathbb{Z})$ such that

$$\widetilde{A} := S \cdot A \cdot T$$

is in Smith normal form; i.e., $\widetilde{A}$ is a "diagonal" matrix of the form

$$\widetilde{A} = \begin{pmatrix} a_1 & & & & & & \\ & \ddots & & & & & \\ & & a_r & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 \end{pmatrix} \in M_{k \times m}(\mathbb{Z})$$

with $r \in \{0, \ldots, \min(m, k)\}$ and $a_1, \ldots, a_r \in \mathbb{Z} \setminus \{0\}$ satisfying $a_1 \mid a_2$, $a_2 \mid a_3$, ..., $a_{r-1} \mid a_r$.

If R is a domain, then (we interpret integral matrices in the canonical way as matrices over R and use the fact that invertible integral matrices stay invertible over R)

$$L_A(R) = L_{S^{-1} \cdot \widetilde{A} \cdot T^{-1}}(R) = T \cdot L_{\widetilde{A}}(R).$$

In particular, we have $L_A(R) = \{0\}$ if and only if

$$r = m \quad \text{and} \quad \forall_{j \in \{1, \ldots, r\}} \quad a_j \neq 0 \text{ in } R.$$

We now consider the set $Q \subset \mathbb{N}$ of prime divisors of $a_1, \ldots, a_r$. Clearly, this set Q is finite. Then $P' := P \setminus Q$ is cofinite in P and for all $p \in P'$ we have $L_{\widetilde{A}}(\mathbb{F}_p) = \pi_p(L_{\widetilde{A}}(\mathbb{Z}))$. Hence, we obtain

$$\begin{aligned} L_A(\mathbb{F}_p) &= T \cdot L_{\widetilde{A}}(\mathbb{F}_p) = T \cdot \pi_p\big(L_{\widetilde{A}}(\mathbb{Z})\big) = \pi_p\big(T \cdot L_{\widetilde{A}}(\mathbb{Z})\big) \\ &= \pi_p\big(L_A(\mathbb{Z})\big) \end{aligned}$$

for all $p \in P'$. In particular, $L_A(\mathbb{Z}) \neq \{0\}$. $\square$

3.4. **Equal characteristic.** As a warm-up for the proof of Theorem 1.2, we prove the following comparison result in equal characteristic:

Theorem 3.8 (equal characteristic)**.** *Let M be an oriented closed connected manifold and let F and E be fields of the same characteristic. Then*

$$\|M\|_{(F)} = \|M\|_{(E)}.$$

Proof. Let $n := \dim M$ and let $c = \sum_{j=1}^m a_j \cdot \sigma_j \in C_n(M; E)$ be an E-fundamental cycle of M in reduced form with $|c| = m = \|M\|_{(E)}$. We consider the model Z of c and the associated cycle matrix $A \in M_{k \times m}(\mathbb{Z})$ (for simplicity, we index the rows of A by natural numbers instead of by pairs). So far, A only encodes the fact that c is a cycle. In order to also encode the corresponding homology class, we proceed as follows:

We pick a point $x \in M$ and consider the local degrees $d_j \in \mathbb{Z}$ determined uniquely by the relation

$$H_n(\sigma_j; \mathbb{Z})[\Delta^n, \partial \Delta^n] = d_j \cdot [M; x]_{\mathbb{Z}} \in H_n\big(M, M \setminus \{x\}; \mathbb{Z}\big);$$

here, $[M; x]_{\mathbb{Z}} = H_n(i; \mathbb{Z})[M]_{\mathbb{Z}}$ denotes the generator in $H_n(M, M \setminus \{x\}; \mathbb{Z}) \cong H_n(M; \mathbb{Z}) \cong \mathbb{Z}$ corresponding to the fundamental class $[M]_{\mathbb{Z}}$ under the inclusion $i \colon (M, \emptyset) \longrightarrow (M, M \setminus \{x\})$. Because c is a fundamental cycle, we obtain

$$\sum_{j=1}^{m} d_j \cdot a_j = 1 \quad \text{in } E.$$

Hence, the coefficients $(a_1, \ldots, a_m) \in E^m$ of c lie in the solution set $L_{\overline{A}, \overline{b}}(E)$, where $\overline{A} \in M_{(k+1) \times m}(\mathbb{Z})$ is the matrix obtained from A by adding the last row $(d_1, \ldots, d_m)$ and where $\overline{b} := (0, \ldots, 0, 1) \in \mathbb{Z}^{k+1}$.

In particular, $L_{\overline{A}, \overline{b}}(E) \neq \emptyset$. Because F and E have the same characteristic, we also have $L_{\overline{A}, \overline{b}}(F) \neq \emptyset$ (Lemma 3.6). Let $(a'_1, \ldots, a'_m) \in L_{\overline{A}, \overline{b}}(F)$. We now consider the chain

$$c' := \sum_{j=1}^{m} a'_j \cdot \sigma_j \in C_n(M; F).$$

By construction, the model of c' is contained in the model Z of c. The chain c' is a cycle because $(a'_1, \ldots, a'_m) \in L_{\overline{A}, \overline{b}}(F) \subset L_A(F)$ and A is the cycle matrix of the model Z (Lemma 3.4). Moreover, c' is an F-fundamental cycle of M: The additional equation

$$\sum_{j=1}^{m} d_j \cdot a'_j = 1 \quad \text{in } F$$

ensures that

$$H_n(i; F)[c'] = \sum_{j=1}^{m} a'_j \cdot H_n(\sigma_j; F)[\Delta^n, \partial \Delta^n]_F = \sum_{j=1}^{m} a'_j \cdot d_j \cdot [M; x]_F$$
$$= [M; x]_F \quad \text{in } H_n(M, M \setminus \{x\}; F),$$

and hence $[c'] = [M]_F$.

In particular, $\|M\|_{(F)} \leq |c'| \leq m = \|M\|_{(E)}$. By symmetry, we also obtain the reverse inequality $\|M\|_{(E)} \leq \|M\|_{(F)}$. $\qquad\square$

3.5. **Comparison with characteristic** 0. Using the tools and techniques from the previous sections, we will now give a proof of Theorem 1.2. The situation in this proof is slightly different from the equal characteristic case because we will need to deal with cycles that have isomorphic models but consist of different singular simplices.

Proof of Theorem 1.2. We begin with the proof that $\|M\|_{(\mathbb{F}_p)} \leq \|M\|_{(\mathbb{Q})}$ holds for all but finitely many primes p: Let $c \in C_n(M; \mathbb{Q})$ be a $\mathbb{Q}$-fundamental cycle of M with $|c| = \|M\|_{(\mathbb{Q})}$. Then there exists an $m \in \mathbb{N} \setminus \{0\}$ such that $m \cdot c$ is an integral cycle, representing $m \cdot [M]_{\mathbb{Z}}$ in $H_n(M; \mathbb{Z})$. Let $P \subset \mathbb{N}$ be the set of primes that do *not* divide m. Then P contains all but finitely many primes in $\mathbb{N}$. Let $p \in P$ and let $c_p \in C_n(M; \mathbb{F}_p)$ be the mod p reduction of c. Then c_p is a cycle, m is a unit modulo p, and

$$\frac{1}{m} \cdot c_p \in C_n(M; \mathbb{F}_p)$$

is an $\mathbb{F}_p$-fundamental cycle of M. In particular,

$$\|M\|_{(\mathbb{F}_p)} \leq \left| \frac{1}{m} \cdot c_p \right| \leq |c| = \|M\|_{(\mathbb{Q})}.$$

We will now prove the converse estimate: By Remark 2.2, we have

$$\left\{ \|M\|_{(\mathbb{F}_p)} \,\big|\, p \in \mathbb{N} \text{ prime} \right\} \subset \left\{ 0, \ldots, \|M\|_{\mathbb{Z}} \right\};$$

in particular, the set on the left hand side is finite. Let $V \in \{0, \ldots, \|M\|_{\mathbb{Z}}\}$ be the smallest accumulation point of $(\|M\|_{(\mathbb{F}_p)})_{p \in \mathbb{N} \text{ prime}}$ and let $P \subset \mathbb{N}$ be the set of primes $p \in \mathbb{N}$ with $V \leq \|M\|_{(\mathbb{F}_p)}$. Because $\{0, \ldots, \|M\|_{\mathbb{Z}}\}$ is finite, the set P is cofinite in the set of primes. Therefore, it suffices to prove that for all but finitely many $p \in P$ we have

$$\|M\|_{(\mathbb{Q})} \leq V \leq \|M\|_{(\mathbb{F}_p)}.$$

Because there exist only finitely many isomorphism classes of n-dimensional model complexes with at most V simplices, there exists such an n-dimensional model complex Z and an infinite subset $P' \subset P$ such that for every $p \in P'$ there exists an $\mathbb{F}_p$-fundamental cycle of M whose model is isomorphic to Z. We will now show that there also exists a $\mathbb{Q}$-fundamental cycle of M whose model is isomorphic to Z:

Let $A \in M_{k \times m}(\mathbb{Z})$ be the integral cycle matrix associated with Z (again, we simplify the index set). By Lemma 3.4, we hence know that

$$\forall_{p \in P'} \quad L_A(\mathbb{F}_p) \neq \{0\}.$$

Applying Lemma 3.7 therefore shows that there exists a cofinite set $P'' \subset P'$ with

$$\forall_{p \in P''} \quad \pi_p\big(L_A(\mathbb{Z})\big) = L_A(\mathbb{F}_p).$$

Let $p \in P''$ and let $c_p = \sum_{j=1}^{V} a_j \cdot \sigma_j \in C_n(M; \mathbb{F}_p)$ be an $\mathbb{F}_p$-fundamental cycle of M with model isomorphic to Z. Because of $\pi_p(L_A(\mathbb{Z})) = L_A(\mathbb{F}_p)$ and Lemma 3.4 there exists a *cycle* $c = \sum_{j=1}^{V} \widetilde{a}_j \cdot \sigma_j \in C_n(M; \mathbb{Z})$ whose reduction modulo p equals c_p and whose model is isomorphic to Z. In particular,

$$[c] \neq 0 \in H_n(M; \mathbb{Z}) \subset H_n(M; \mathbb{Q})$$

(because its reduction $[c_p]$ is non-zero in $H_n(M; \mathbb{F}_p)$). Then a rational multiple of c is a $\mathbb{Q}$-fundamental cycle of M and we obtain $\|M\|_{(\mathbb{Q})} \leq |c| \leq V$. $\square$

Corollary 3.9. *Let M be an oriented closed connected manifold and let F be a field of characteristic 0. Then for all but finitely many primes $p \in \mathbb{N}$ we have*

$$\|M\|_{(\mathbb{F}_p)} = \|M\|_{(F)}.$$

Proof. We only need to combine Theorem 1.2 with Theorem 3.8. $\square$

4. STABILISATION ALONG FINITE COVERINGS

In this section, we discuss the stabilisation of weightless simplicial volumes along towers of finite coverings. In particular, we will prove Theorem 1.7 and Theorem 1.8.

4.1. **Stable weightless simplicial volumes.** The classical simplicial volume $\| \cdot \|_{\mathbb{R}}$ is multiplicative under finite coverings; however, integral simplicial volume $\| \cdot \|_{\mathbb{Z}}$, in general, is *not* multiplicative under finite coverings. Hence, it makes sense to study the corresponding gradient invariant, the *stable integral simplicial volume*, defined by

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf\Big\{ \frac{\|N\|_{\mathbb{Z}}}{d} \,\Big|\, d \in \mathbb{N},\, N \to M \text{ is a } d\text{-sheeted covering}\Big\}$$

for oriented closed connected n-manifolds M.

Analogously, we can also introduce the gradient invariants associated with weigthless simplicial volumes:

Definition 4.1 (stable weightless simplicial volume)**.** Let M be an oriented closed connected n-manifold and let R be a commutative ring with unit. Then the *stable weightless R-simplicial volume* of M is defined by

$$\|M\|_{(R)}^{\infty} := \inf\Big\{ \frac{\|N\|_{(R)}}{d} \,\Big|\, d \in \mathbb{N},\, N \to M \text{ is a } d\text{-sheeted covering}\Big\}.$$

This generalises Definition 1.5.

Remark 4.2 (basic estimates)**.** Let M be an oriented closed connected n-manifold and let R be a commutative ring with unit.

(1) Then $\|M\|_{\mathbb{R}} \leq \|M\|_{\mathbb{Z}}^{\infty}$ and $\|M\|_{(R)}^{\infty} \leq \|M\|_{\mathbb{Z}}^{\infty}$.

(2) If M admits a self-covering of non-trivial degree, then $\|M\|_{(R)}^{\infty} = 0$. In particular, the stable weightless simplicial volumes of tori are zero.

(3) If $(\Gamma_j)_{j\in\mathbb{N}}$ is a descending chain of finite index subgroups of $\pi_1(M)$ and $(M_j)_{j\in\mathbb{N}}$ is the corresponding tower of finite covering manifolds of M, then the sequence

$$\Big(\frac{1}{[\pi_1(M) : \Gamma_j]} \cdot \|M_j\|_{(R)} \Big)_{j\in\mathbb{N}}$$

is monotonically decreasing (full lifts of fundamental cycles give fundamental cycles of finite coverings), whence convergent. Therefore,

$$\lim_{j\to\infty} \frac{\|M_j\|_{(R)}}{[\pi_1(M) : \Gamma_j]} = \inf_{j\to\infty} \frac{\|M_j\|_{(R)}}{[\pi_1(M) : \Gamma_j]}.$$

(4) Every (generalised) triangulation of M gives rise to an $\mathbb{F}_2$-fundamental cycle. Hence, we obtain

$$\|M\|_{(\mathbb{F}_2)}^{\infty} \leq \sigma_{\infty}(M),$$

where $\sigma_{\infty}(M)$ denotes the stable Δ-complexity of M [12].

4.2. **The Euler characteristic estimate.** We start with a few simple example estimates for stable weightless simplicial volumes of even-dimensional hyperbolic manifolds.

Example 4.3 (surfaces)**.** Let Σ be an oriented closed connected (non-empty) surface of genus $g \in \mathbb{N}_{\geq 1}$ and let $p \in \mathbb{N}$ be a prime. Then we know $2 \cdot g \leq \|\Sigma\|_{(\mathbb{F}_p)} \leq 4 \cdot g - 2$ (Example 2.12). Taking the infimum over all finite coverings of Σ produces the estimates (and similarly for $\|M\|_{(\mathbb{Z})}^{\infty}$)

$$\big|\chi(\Sigma)\big| = 2 \cdot g - 2 \leq \|\Sigma\|_{(\mathbb{F}_p)}^{\infty} \leq 4 \cdot g - 4 = \|\Sigma\|_{\mathbb{R}} = 2 \cdot \big|\chi(\Sigma)\big|.$$

However, the exact values of $\|\Sigma\|^\infty_{(\mathbb{F}_p)}$ or $\|\Sigma\|^\infty_{(\mathbb{Z})}$ are not known if $g \geq 2$.

Proposition 4.4 (Euler characteristic estimate)**.** *Let M be an oriented closed connected n-manifold and let R be a principal ideal domain. Then*

$$\big|\chi(M)\big| \leq (n+1) \cdot \|M\|_{(R)}.$$

In particular: For all prime numbers $p \in \mathbb{N}$ we have $\big|\chi(M)\big| \leq (n+1) \cdot \|M\|_{(\mathbb{F}_p)}$.

Proof. Let $N \longrightarrow M$ be a finite covering of M with $d \in \mathbb{N}$ sheets. Then the Betti number estimate of Proposition 2.6 shows that

$$\big|\chi(M)\big| = \frac{1}{d} \cdot \big|\chi(N)\big| \leq \frac{1}{d} \cdot \sum_{j=0}^n \operatorname{rk}_R H_j(N; R)$$

$$\leq (n+1) \cdot \frac{1}{d} \cdot \|N\|_{(R)}.$$

Taking the infimum over all finite coverings of M finishes the proof. $\square$

Corollary 4.5. *Let $n \in \mathbb{N}$ be even. Then there exists a constant $C_n \in \mathbb{N}_{>0}$ such that: For every oriented closed connected hyperbolic n-manifold and every principal ideal domain R we have*

$$\|M\|_{(R)} \geq C_n \cdot \operatorname{vol}(M).$$

In particular: For all prime numbers $p \in \mathbb{N}$ we have $\|M\|_{(\mathbb{F}_p)} \geq C_n \cdot \operatorname{vol}(M)$.

Proof. Applying the generalised Gauß-Bonnet formula [13, Theorem 11.3.2] to the hyperbolic manifold M results in

$$\chi(M) = (-1)^{n/2} \cdot \frac{2}{\Omega_n} \cdot \operatorname{vol}(M),$$

where Ω_n denotes the volume of the standard unit n-sphere.

In combination with Proposition 4.4, we therefore obtain

$$\|M\|^\infty_{(R)} \geq \frac{1}{n+1} \cdot \big|\chi(M)\big| = \frac{2}{(n+1) \cdot \Omega_n} \cdot \operatorname{vol}(M). \qquad \square$$

4.3. **Self-maps of non-trivial degree.** We will now prove Theorem 1.7 in the following, slightly more general, form:

Theorem 4.6. *Let M be an oriented closed connected aspherical manifold with residually finite fundamental group that admits a continuous self-map $f\colon M \longrightarrow M$ with $\deg f \notin \{-1, 0, 1\}$. If R is a commutative ring with unit and $\deg f$ is a unit in R, then*

$$\|M\|^\infty_{(R)} = 0.$$

We proceed in the same way as in the proof of the corresponding statement for L^2-Betti numbers [11, Theorem 14.40]. As a preparation, we recall the following well-known fact:

Lemma 4.7 (mapping degrees and index of subgroups)**.**

(1) *Let M and N be oriented closed connected manifolds of the same dimension and let $f\colon M \longrightarrow N$ be a continuous map of non-zero degree. Then $\operatorname{im} \pi_1(f)$ has finite index in $\pi_1(N)$.*

(2) *Let M be an oriented closed connected aspherical manifold with residually finite fundamental group and let $f\colon M \longrightarrow M$ be a continuous map with $\deg f \notin \{-1, 0, 1\}$. Then*

$$1 < \big[\pi_1(M) : \operatorname{im}\pi_1(f)\big] < \infty.$$

Proof. The first part follows from covering theory (by considering the covering of N associated with $\operatorname{im}\pi_1(f)$).

For the second part, we write $\Gamma := \pi_1(M)$. By the first part, $\operatorname{im}\pi_1(f)$ has finite index in Γ.

Assume for a contradiction that $[\Gamma : \operatorname{im}\pi_1(f)] = 1$, i.e., that $\pi_1(f)$ is an epimorphism. As residually finite group, the group Γ is Hopfian. Hence, the epimorphism $\pi_1(f)\colon \Gamma \longrightarrow \Gamma$ is an isomorphism. Because M is aspherical, this implies that f is a homotopy equivalence. In particular, $\deg f \in \{-1, 1\}$, which is a contradiction. Therefore, $[\Gamma : \operatorname{im}\pi_1(f)] > 1$. $\square$

Proof of Theorem 4.6. Let $\Gamma := \pi_1(M)$. For $k \in \mathbb{N}$ let $f_k := f^{\circ k}\colon M \longrightarrow M$ be the k-fold composition of f and let $\Gamma_k := \operatorname{im}\pi_1(f_k) \subset \Gamma$ be the corresponding subgroup. For $k \in \mathbb{N}$, let $p_k\colon M_k \longrightarrow M$ be the covering associated with the subgroup $\Gamma_k \subset \Gamma$; thus, there is a p_k-lift $\overline{f}_k\colon M \longrightarrow M_k$ of f_k. By construction,

$$\deg \overline{f}_k \cdot \deg p_k = \deg(p_k \circ \overline{f}_k) = \deg f_k$$
$$= (\deg f)^k,$$

and so $\deg \overline{f}_k$ is a unit in R. In view of Proposition 2.3, we obtain

$$\|M_k\|_{(R)} \leq \|M\|_{(R)};$$

thus,

$$\|M\|_{(R)}^\infty \leq \inf_{k \in \mathbb{N}} \frac{\|M_k\|_{(R)}}{|\deg p_k|} \leq \inf_{k \in \mathbb{N}} \frac{\|M\|_{(R)}}{[\Gamma : \Gamma_k]}.$$

Therefore, it suffices to show that the sequence $([\Gamma : \Gamma_k])_{k \in \mathbb{N}}$ is unbounded: To this end, for $k \in \mathbb{N}$, we consider the self-map

$$g_k := \overline{f}_k \circ p_k\colon M_k \longrightarrow M_k$$

of M_k. Because Γ_k has finite index in Γ, the covering manifold M_k is compact; morevoer, M_k is aspherical and oriented and $\pi_1(M_k) \cong \Gamma_k \subset \Gamma$ is residually finite. Using the fact that

$$\deg g_k = \deg \overline{f}_k \cdot \deg p_k = (\deg f)^k \notin \{-1, 0, 1\},$$

we obtain from Lemma 4.7 that

$$1 < \big[\pi_1(M_k) : \operatorname{im}\pi_1(g_k)\big] = \big[\Gamma_k : \pi_1(f_k)(\Gamma_k)\big] = [\Gamma_k : \Gamma_{2\cdot k}].$$

In particular, the sequence $([\Gamma : \Gamma_{2^k}])_{k \in \mathbb{N}}$ is unbounded. $\square$

Proof of Theorem 1.7. If $p \nmid \deg f$, then $\deg f$ is a unit in $\mathbb{F}_p$ and so Theorem 4.6 can be applied. $\square$

4.4. **The dynamical view.** In order to formulate and prove Theorem 1.8, we first introduce the type of dynamical systems that we are interested in:

Definition 4.8. Let Γ be a group. A *standard Γ-space* is a standard Borel probability space (X, μ) together with a μ-preserving Γ-action.

If $\alpha = \Gamma \curvearrowright (X, \mu)$ is a standard Γ-space and R is a commutative ring with unit, then we write

$$L^\infty(\alpha, R) := L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} R.$$

This module is equipped with the $\mathbb{Z}\Gamma$-right module structure induced by the Γ-action α on X. More concretely, $L^\infty(\alpha, R)$ can be viewed as the $\mathbb{Z}\Gamma$-module of functions $X \longrightarrow R$ with finite image and measurable pre-images (up to equality μ-almost everywhere).

If $f \colon X \longrightarrow R$ is an element of $L^\infty(\alpha, R)$, then we write

$$\operatorname{supp} f := f^{-1}(R \setminus \{0\}) \subset X.$$

Example 4.9 (profinite completion). If Γ is a finitely generated residually finite group, then the profinite completion $\widehat{\Gamma}$ of Γ is a standard Borel space. The canonical translation action of Γ on $\widehat{\Gamma}$ is measure preserving with respect to the inverse limit probability measure of the normalised counting measures on the finite quotients of Γ. Moreover, this action is essentially free. For simplicity, we also denote the corresponding standard Γ-space by $\widehat{\Gamma}$.

In analogy with parametrised/integral foliated simplicial volume [15, 10], we consider weightless parametrised simplicial volumes (by ignoring the magnitude of the coefficients):

Definition 4.10 (weightless parametrised simplicial volume). Let M be an oriented closed connected n-manifold, let $\alpha := \pi_1(M) \curvearrowright (X, \mu)$ be a standard $\pi_1(M)$-space, and let R be a commutative ring with unit. A cycle $c \in C_n(M; L^\infty(\alpha, R)) = L^\infty(X, R) \otimes_{\mathbb{Z}\pi_1(M)} C_n(\widetilde{M}; \mathbb{Z})$ is an $(\alpha; R)$-*fundamental cycle of M* if c is homologous (in $C_*(M; L^\infty(\alpha, R))$) to an integral fundamental cycle of M (via the canonical map $C_*(M; \mathbb{Z}) \longrightarrow C_*(M; L^\infty(\alpha, R))$ given by the constant functions). The *weightless parametrised R-simplicial volume* of M is defined as

$$\|M\|_{(\alpha;R)} := \inf\left\{ \sum_{j=1}^m \mu(\operatorname{supp} f_j) \;\middle|\; \sum_{j=1}^m f_j \otimes \sigma_j \in C_n\big(M; L^\infty(\alpha, R)\big) \right.$$
$$\left. \text{is an } (\alpha; R)\text{-fundamental cycle of } M \right\}.$$

Clearly, in the situation of the previous definition, we have

$$0 \le \|M\|_{(\alpha;R)} \le \|M\|_{(\alpha;\mathbb{Z})} \le |M|^\alpha,$$

where $|M|^\alpha$ denotes the ordinary parametrised simplicial volume with parameter space α.

Finally, we prove Theorem 1.8; again, we establish a slightly more general version:

Theorem 4.11. *Let M be an oriented closed connected manifold with residually finite fundamental group and let R be a commutative ring with unit. Then*

$$\|M\|_{(R)}^{\infty} = \|M\|_{(\widehat{\pi_1(M)};R)}.$$

The proof is a straightforward adaption of the proof of the corresponding statement for stable integral simplicial volume [10, Theorem 6.6, Remark 6.7]. We first set up some notation for the proof:

- We abbreviate $\Gamma := \pi_1(M)$ and $\alpha := \Gamma \curvearrowright \widehat{\Gamma}$.
- If $\Lambda \subset \Gamma$ is a finite index subgroup, we write $M_\Lambda \longrightarrow M$ for the associated covering.
- We write S for the set of all finite index subgroups and $F(S)$ for the set of all finite subsets of S.
- If $F \in F(S)$, then we write $X_F := \varprojlim_{\Lambda \in F} \Gamma/\Lambda$ and $\alpha_F := \Gamma \curvearrowright X_F$ for the associated parameter space. Moreover, we denote the canonical Γ-map $\widehat{\Gamma} \longrightarrow X_F$ by π_F.

In the proof of Theorem 4.11, we will use the following ingredients:

(1) If Λ is a finite index subgroup of Γ, then

$$\|M\|_{(\alpha_{\{\Lambda\}};R)} = \frac{1}{[\Gamma : \Lambda]} \cdot \|M_\Lambda\|_{(R)}.$$

[The proof of the corresponding statement for parametrised integral simplicial volume [10, Proposition 4.26, Corollary 4.27] carries over to the weightless setting, because the division by the index happens inside the probability space.]

(2) If $F \in F(S)$, then α_F is a finite Γ-probability space and thus a finite convex combination of coset spaces as in (1). Therefore,

$$\|M\|_{(R)}^{\infty} \leq \|M\|_{(\alpha_F;R)}.$$

[Again, the arguments of the classical case [10, Proposition 4.15] also work in the weightless setting.]

(3) Let $L \subset L^\infty(\alpha, R)$ be a $\mathbb{Z}\Gamma$-submodule that is dense in the following sense:

$$\forall_{f \in L^\infty(\alpha, R)} \quad \forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{g \in L} \quad \mu\big(\mathrm{supp}(f - g)\big) < \varepsilon.$$

Then the induced homomorphism $H_n(M; L) \longrightarrow H_n(M; L^\infty(\alpha, R))$ is norm-preserving with respect to the weightless norms.
[The standard proof [15, Lemma 2.9][7, Proposition 1.7] of statements of this type – by approximating boundaries – also works in the weightless setting.]

Proof of Theorem 4.11. We first prove that $\|M\|_{(\alpha,R)} \leq \|M\|_{(R)}^{\infty}$: If $\Lambda \in S$, then the canonical projection $\alpha \longrightarrow \alpha_{\{\Lambda\}}$ and (1) show that

$$\|M\|_{(\alpha;R)} \leq \|M\|_{(\alpha_{\{\Lambda\}};R)} = \frac{1}{[\Gamma : \Lambda]} \cdot \|M_\Lambda\|_{(R)}.$$

Every (connected) finite covering of M corresponds to a finite index subgroup of $\pi_1(M)$. Therefore, taking the infimum over all Λ in S implies

$$\|M\|_{(\alpha;R)} \leq \inf_{\Lambda \in S} \frac{1}{[\Gamma : \Lambda]} \cdot \|M_\Lambda\|_{(R)} = \|M\|_{(R)}^\infty.$$

Conversely, we now show that $\|M\|_{(R)}^\infty \leq \|M\|_{(\alpha,R)}$: To this end we consider the $\mathbb{Z}\Gamma$-submodule

$$L := \bigcup_{F \in F(S)} \{ f \circ \pi_F \mid f \in L^\infty(\alpha_F, R) \}$$

of $L^\infty(\alpha, R)$. The submodule L is dense in $L^\infty(\alpha, R)$ in the sense of (3): Let σ be the σ-algebra of $\widehat{\Gamma}$. The set

$$\sigma' := \{ A \in \sigma \mid \forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{B \in \sigma} \quad \chi_B \in L \wedge \mu(A \triangle B) < \varepsilon \}$$

is a σ-algebra. Moreover, σ' is contained in σ and σ' contains for every $F \in F(S)$ the π_F-preimage of the σ-algebra of X_F. Hence, $\sigma' = \sigma$ and so L is dense in $L^\infty(\alpha, R)$.

Combining (2), (3), and the construction of L, we obtain

$$\|M\|_{(R)}^\infty \leq \inf_{F \in F(S)} \|M\|_{(\alpha_F;R)} = \|M\|_{(\alpha;R)}. \qquad \square$$

Example 4.12 (small amenable covers). Let M be an oriented closed connected aspherical (triangulable) n-manifold with residually finite fundamental group that admits an open cover by amenable sets such that each point of M is contained in at most n of these sets. Then

$$\|M\|_{(R)}^\infty = 0$$

for all commutative rings R with unit (in particular, $\|M\|_{(\mathbb{F}_p)}^\infty = 0$ for all prime numbers p). Before giving a proof of this statement, we recall that a subset $U \subset M$ of M is *amenable* if for every $x \in U$ the image of the homomorphism $\pi_1(U, x) \longrightarrow \pi_1(M, x)$ induced by the inclusion is amenable.

Indeed, by work of Sauer [14, proof of Theorem B, Section 5.3] we have

$$\|M\|_{(\widehat{\pi_1(M)};\mathbb{Z})} = 0$$

(Sauer's notion of mass of the fundamental class of M with respect to the action of $\pi_1(M)$ on $\widehat{\pi_1(M)}$ coincides with $\|M\|_{(\widehat{\pi_1(M)};\mathbb{Z})}$). Therefore, the canonical ring homomorphism $\mathbb{Z} \longrightarrow R$ shows that

$$0 \leq \|M\|_{(\widehat{\pi_1(M)};R)} \leq \|M\|_{(\widehat{\pi_1(M)};\mathbb{Z})} = 0.$$

Applying Theorem 4.11, we hence obtain

$$\|M\|_{(R)}^\infty = \|M\|_{(\widehat{\pi_1(M)};R)} = 0.$$

References

[1] R. Benedetti, C. Petronio. *Lectures on Hyperbolic Geometry*, Universitext, Springer, 1992. Cited on page: 8

[2] D. Fauser. Integral foliated simplicial volume and S^1-actions, preprint, available at arXiv:1704.08538 [math.GT], 2017. Cited on page: 3

[3] D. Fauser, S. Friedl, C. Löh. Integral approximation of simplicial volume of graph manifolds, preprint, available at arXiv:1807.10522 [math.GT], 2018. To appear in *Bull. London Math. Soc.*. Cited on page: 3

[4] R. Frigerio, C. Löh, C. Pagliantini, R. Sauer. Integral foliated simplicial volume of aspherical manifolds, *Israel J. Math.*, 216(2), 707–751, 2016. Cited on page: 3

[5] M. Gromov. Volume and bounded cohomology, *Inst. Hautes Études Sci. Publ. Math.*, 56, 5–99, 1983. Cited on page: 1, 2, 5, 8

[6] M. Gromov. *Asymptotic invariants of infinite groups.* Geometric group theory, Vol. 2 (Sussex 1991). London Math. Soc. Lectures Notes Ser., 182, Cambridge University Press, Cambridge, 1–295, 1993. Cited on page: 4

[7] C. Löh. ℓ^1-*Homology and Simplicial Volume*, PhD thesis, Westfälische Wilhelms-Universität Münster, 2007.
http://nbn-resolving.de/urn:nbn:de:hbz:6-37549578216 Cited on page: 19

[8] C. Löh. Simplicial Volume, *Bull. Man. Atl.*, 7–18, 2011. Cited on page: 1

[9] C. Löh. Odd manifolds of small integral simplicial volume, *Arkiv för Matematik*, 56(2), 351–375, 2018. Cited on page: 7, 8, 9

[10] C. Löh, C. Pagliantini. Integral foliated simplicial volume of hyperbolic 3-manifolds, *Groups Geom. Dyn.*, 10(3), 825–865, 2016. Cited on page: 4, 18, 19

[11] W. Lück. L^2-*Invariants: Theory and Applications to Geometry and K-Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3. Folge, 44. Springer, 2002. Cited on page: 3, 16

[12] J. Milnor, W. Thurston. Characteristic numbers of 3-manifolds, *Enseignement Math. (2)*, 23(3–4), 249–254, 1977. Cited on page: 15

[13] J.G. Ratcliffe. *Foundations of Hyperbolic Manifolds*. Graduate Texts in Mathematics, 149, Springer, 1994. Cited on page: 16

[14] R. Sauer. Amenable covers, volume and L^2-Betti numbers of aspherical manifolds, *J. Reine Angew. Math (Crelle's Journal)*, 636, 47–92, 2009. Cited on page: 3, 20

[15] M. Schmidt. L^2-*Betti Numbers of $\mathcal{R}$-spaces and the Integral Foliated Simplicial Volume*. PhD thesis, Westfälische Wilhelms-Universität Münster, 2005.
http://nbn-resolving.de/urn:nbn:de:hbz:6-05699458563 Cited on page: 18, 19

Clara Löh

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg
clara.loeh@mathematik.uni-r.de,
http://www.mathematik.uni-r.de/loeh