FUNCTORIAL SEMI-NORMS ON SINGULAR HOMOLOGY AND (IN)FLEXIBLE MANIFOLDS

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ABSTRACT. A functorial semi-norm on singular homology is a collection of semi-norms on the singular homology groups of spaces such that continuous maps between spaces induce norm-decreasing maps in homology. Functorial semi-norms can be used to give constraints on the possible mapping degrees of maps between oriented manifolds.

In this paper, we use information about the degrees of maps between manifolds to construct new functorial semi-norms with interesting properties. In particular, we answer a question of Gromov by providing a functorial semi-norm that takes finite positive values on homology classes of certain *simply connected* spaces. Our construction relies on the existence of simply connected manifolds that are *inflexible* in the sense that all their self-maps have degree -1, 0, or 1. The existence of such manifolds was first established by Arkowitz and Lupton; we extend their methods to produce a wide variety of such manifolds.

1. INTRODUCTION

Enriching algebraic invariants with metric data is a common theme in many branches of mathematics. Gromov introduced the concept of functorial semi-norms on singular homology [11, Section 5.34], which are an example of this paradigm in topology.

A *functorial semi-norm* on singular homology consists of the addition of a semi-normed structure to the singular homology groups with $\mathbb{R}$-coefficients in such a way that continuous maps induce linear maps on homology of norm at most 1 (Definition 2.1). An interesting aspect is that suitable functorial semi-norms give a systematic way to deduce degree theorems for maps between manifolds (Remark 2.6). Conversely, in the present paper, we translate knowledge about degrees of maps between manifolds to construct new functorial semi-norms.

A central example of a functorial semi-norm on singular homology, studied by Gromov [8], is the ℓ^1-semi-norm given by taking the infimum of the ℓ^1-norms of all cycles representing a given homology class (Example 2.2). The ℓ^1-semi-norm gives rise to lower bounds for the minimal volume and hence leads to interesting applications in Riemannian geometry [8]. On the other hand, using bounded cohomology, Gromov showed that the ℓ^1-semi-norm vanishes on classes of non-zero degree of simply connected spaces [8], and later raised the question whether every functorial semi-norm on singular homology in non-zero degree is trivial on all simply connected spaces [11, Remark (b) in 5.35]. More precisely, we formulate this problem as follows:

Date: March 10, 2015.

MSC 2010 classification: primary:57N65, secondary: 55N10, 55N35, 55P62.

Question 1.1. *Let $d \in \mathbb{N}_{>0}$.*

(1) *Does every (possibly infinite) functorial semi-norm on singular homology in degree d take only the values 0 and ∞ on homology classes of simply connected spaces?*
(2) *Does every finite functorial semi-norm on singular homology in degree d vanish on homology classes of simply connected spaces?*

In this paper, we answer the first part of this question in the negative (Corollary 7.4):

Theorem 1.2. *There are functorial semi-norms on singular homology that are positive and finite on certain homology classes of simply connected spaces.*

More concretely, we give examples of such functorial semi-norms in all degrees in the set $\{64\} \cup \{d \cdot k \mid k \in \mathbb{N}_{>0},\ d \in \{108, 208, 228\}\}$ (Corollary 9.7).

On the other hand, we give a positive answer to Question 1.1(2) in low dimensions (Section 7.2):

Theorem 1.3. *All finite functorial semi-norms on singular homology in the degrees $1, \dots, 6$ vanish on all homology clases of simply connected spaces.*

The key to proving Theorem 1.2 and 1.3 is to gain an understanding of the class of simply connected inflexible manifolds.

Definition 1.4 (Inflexible manifolds)**.** If M and N are oriented closed connected manifolds of the same dimension, then we write

$$\deg(N, M) := \{\deg f \mid f \colon N \longrightarrow M \text{ continuous}\}$$

for the set of all possible mapping degrees for maps from N to M. An oriented closed connected manifold M is *inflexible* if $\deg(M, M) \subset \{-1, 0, 1\}$.

The proof of Theorem 1.2 consists of two main steps:

– *Generating functorial semi-norms via manifolds.* Using the fact that singular homology classes can (up to a scalar multiple) be represented by fundamental classes of oriented closed connected manifolds (Section 3), we show how functorial semi-norms on fundamental classes of manifolds of a given dimension can be extended to functorial semi-norms on singular homology (Theorem 4.2).
– *Inflexible manifolds.* With the help of simply connected inflexible manifolds, we construct a functorial semi-norm on fundamental classes of manifolds that is positive and finite on the given simply connected inflexible manifold (Corollary 7.4).

Simply connected inflexible manifolds can be constructed by means of rational homotopy theory and surgery theory. The first examples of such manifolds were given by Arkowitz and Lupton [2, Examples 5.1 and 5.2]; these examples have dimension 208 and 228 respectively. Using and extending the methods of Arkowitz and Lupton, we give more examples of simply connected inflexible manifolds: For instance, we have examples in dimension 64 (the smallest dimension known before being 208) and 108. Starting from these basic examples, we can construct many more simply connected inflexible manifolds:

– In general, it is not clear that connected sums and products of inflexible manifolds are inflexible; however, in certain cases this is true (Section 9.1 and 9.2). This provides in infinitely many dimensions infinitely many rational homotopy types of oriented closed simply connected inflexible manifolds (Corollary 9.7).
– In addition, using scaling of the fundamental class with respect to a rationalisation, we obtain infinitely many homotopy types of oriented closed simply connected inflexible manifolds within the same rational homotopy type (Proposition 9.8).
– Moreover, we can show that for manifolds being simply connected and inflexible is generic in the sense that in infinitely many dimensions every rational bordism class is represented by a simply connected inflexible manifold (Proposition 9.12).
– Also, there are simply connected inflexible smooth manifolds satisfying certain tangential structure constraints such as being stably parallelisable or non-spinable (Section 9.3).

However, from our construction it is not clear whether the examples from Theorem 1.2 are finite functorial semi-norms; so Gromov's question remains open for finite functorial semi-norms in degree 7 and higher. More precisely, we prove the following proposition (Proposition 7.6) where an oriented closed connected n-manifold M is called *strongly inflexible* if for any oriented closed connected n-manifold N the set $\deg(N, M)$ is finite (Definition 6.14):

Proposition 1.5. *For $d \in \mathbb{N}_{\geq 4}$ the following statements are equivalent:*

(1) *There is a* finite *functorial semi-norm $|\cdot|$ on $H_d(\,\cdot\,;\mathbb{R})$ such that for some homology class $\alpha \in H_d(X;\mathbb{R})$ of some simply connected space X we have $|\alpha| \neq 0$.*
(2) *There exists an oriented closed simply connected d-manifold that is strongly inflexible.*

No example of a simply connected strongly inflexible manifold seems to be known to date: if such a manifold exists, it has dimension at least 7.

Remark 1.6. Since this paper was posted Costoya and Viruel [6] and also Amann [1] have further extended the list of examples and constructions of simply connected inflexible manifolds. Amann [1] has also given new examples of simply connected flexible manifolds.

Organisation of this paper. We start by giving an introduction to functorial semi-norms (Section 2). In Section 3 we recall Thom's result on representation of homology classes by fundamental classes of manifolds, which is the key ingredient for generating functorial semi-norms via functorial semi-norms for manifolds (Section 4). We discuss the relationship between functorial semi-norms on the singular chain complex and functorial semi-norms on singular homology in Section 5. In Section 7 we prove the Theorems 1.2 and 1.3. The proof of Theorem 1.2 is based on the construction of simply connected inflexible manifolds; we carefully review and extend the construction of Arkowitz and Lupton of simply connected inflexible manifolds in Section 6, the technical aspects being deferred to Section 8.

Finally, Section 9 contains the study of inheritance properties of being inflexible and evidence for the genericity of inflexibility in the class of simply connected manifolds.

Acknowledgements. We are indebted to Donald Stanley who drew our attention to the examples of Arkowitz and Lupton. Moreover, we would like to thank Thomas Schick for interesting discussions. We are grateful to Jonathan Bowden for pointing out a mistake in a previous version. Part of this work was supported by the HIM trimester program *Rigidity* and by the SFB 878 *Groups, Geometry and Actions*.

2. FUNCTORIAL SEMI-NORMS

Functorial semi-norms assign a notion of "size" to singular homology classes in a functorial way (Definition 2.1).

In this paper, we use the following convention: A *semi-norm* on an $\mathbb{R}$-vector space V is a function $|\cdot| : V \longrightarrow [0,\infty]$ satisfying the following properties:

- We have $|0| = 0$.
- For all $x \in V$ and all $a \in \mathbb{R} \setminus \{0\}$, we have $|a \cdot x| = |a| \cdot |x|$, where $|a| \cdot \infty := \infty$.
- For all $x, y \in V$ the triangle inequality $|x+y| \leq |x| + |y|$ holds.

A semi-norm is called *finite* if it does not take the value ∞.

Definition 2.1 (Functorial semi-norms [11, Section 5.34])**.** Let $d \in \mathbb{N}$. A *functorial semi-norm (on singular homology) in degree d* consists of a choice of a semi-norm $|\cdot|$ on $H_d(X;\mathbb{R})$ for any topological space X such that the following "functoriality" holds: for all continuous maps $f\colon X \longrightarrow Y$ between topological spaces and all $\alpha \in H_d(X;\mathbb{R})$ we have

$$\bigl|H_d(f;\mathbb{R})(\alpha)\bigr| \leq |\alpha| .$$

Such a functorial semi-norm is called *finite*, if all the semi-norms involved are finite semi-norms.

Example 2.2 (ℓ^1-Semi-norm). For a topological space X let $|\cdot|_1$ denote the ℓ^1-norm on the singular chain complex $C_*(X;\mathbb{R})$ with respect to the (unordered) basis given by all singular simplices: if $c = \sum_{j=1}^k a_j \cdot \sigma_j \in C_*(X;\mathbb{R})$ is in reduced form, then we define

$$|c|_1 := \sum_{j=1}^k |a_j|.$$

This norm induces a finite semi-norm $\|\cdot\|_1$, the so-called *ℓ^1-semi-norm*, on singular homology as follows: for all $\alpha \in H_*(X;\mathbb{R})$ we set

$$\|\alpha\|_1 := \inf\bigl\{|c|_1 \bigm| c \in C_*(X;\mathbb{R}) \text{ is a cycle representing } \alpha\bigr\}.$$

Looking at the definition of the homomorphisms induced by continuous maps in singular homology, it is immediate that $\|\cdot\|_1$ is a functorial semi-norm on singular homology.

An interesting topological invariant derived from the ℓ^1-semi-norm in singular homology is the simplicial volume, introduced by Gromov [11]: If M is an oriented closed connected manifold, then

$$\|M\| := \big\|[M]_{\mathbb{R}}\big\|_1 \in \mathbb{R}_{\geq 0}$$

is the *simplicial volume* of M, where $[M]_{\mathbb{R}} \in H_{\dim M}(M;\mathbb{R})$ denotes the $\mathbb{R}$-fundamental class of M. E.g., using self-maps of non-trivial degree, one sees that the simplicial volume of spheres (of non-zero dimension) is zero. On the other hand, for example, the simplicial volume of oriented closed connected hyperbolic manifolds is non-zero [8, 24, Section 0.3, Theorem 6.2], leading to interesting applications in Riemannian geometry [8].

The ℓ^1-semi-norm on singular homology can also be expressed in terms of bounded cohomology [8, 5, p. 17, Proposition F.2.2]. Using bounded cohomology, Gromov discovered that the ℓ^1-semi-norm of simply connected spaces is trivial [8, 12, Section 3.1, Theorem 2.4], and, more generally, that continuous maps that induce an isomorphism on the level of fundamental groups induce norm-preserving maps on the level of singular homology [8, 12, Section 3.1, Theorem 4.3].

It is tempting to analogously consider ℓ^p-norms with $p > 1$; however, it can be shown that the corresponding definition then leads to the zero semi-norm on homology in positive degrees (this follows from an argument similar to (Non-)Example 5.1)

Example 2.3 (Domination by products of surfaces). For $d \in \mathbb{N}$, we define the functorial semi-norm $|\cdot|_S$ in degree $2d$ as follows [11, Section 5.34]: Let X be a topological space, and let $\alpha \in H_{2d}(X;\mathbb{R})$. Then

$$\begin{aligned} |\alpha|_S := \inf\Big\{ \sum_{j=1}^{k} |a_j| \cdot |\chi(S_j)| \;\Big|\; & k \in \mathbb{N},\ a_1,\dots,a_k \in \mathbb{R}\setminus\{0\}, \\ & S_1,\dots,S_k \text{ are } d\text{-fold products} \\ & \quad \text{of oriented closed connected surfaces}, \\ & f_1\colon S_1 \to X,\dots,f_k\colon S_k \to X \text{ continuous} \\ & \text{with } \sum_{j=1}^{k} a_j \cdot H_{2\cdot d}(f_j;\mathbb{R})[S_j]_{\mathbb{R}} = \alpha \Big\}. \end{aligned}$$

In other words, $|\cdot|_S$ measures the size of homology classes in terms of products of surfaces. In general, this functorial semi-norm is *not* finite [13] because not every homology class in even degree can be represented by a product of surfaces.

Proposition 2.4 (The surface semi-norm is the ℓ^1-semi-norm in degree 2)**.** *Let X be a topological space, and let $\alpha \in H_2(X;\mathbb{R})$. Then*

$$\|\alpha\|_1 = 2 \cdot |\alpha|_S.$$

Proof. This follows from a result of Barge and Ghys [4, Proposition 1.9] (notice that their argument applies only to classes that do not need summands represented by S^2 [4, proof of Lemme 1.7]; however, we can safely ignore these summands as they do not contribute to the ℓ^1-semi-norm). □

Question 2.5. *Does the surface semi-norm vanish on homology classes (of non-zero degree) of simply-connected spaces?*

Classical arguments from algebraic topology show that this is indeed true in degrees 2 and 4 (see Proposition 4.5); however, the question is open in high degrees.

Similarly to the surface semi-norm $\lvert\cdot\rvert_S$, we can also define functorial semi-norms by looking at domination by, e.g., hyperbolic manifolds (Example 4.6).

An interesting aspect of functorial semi-norms is that suitable functorial semi-norms give a systematic way to deduce degree theorems for maps between manifolds:

Remark 2.6 (Degree theorems). If $\lvert\cdot\rvert$ is a functorial semi-norm on singular homology, then by definition we have for all continuous maps $f\colon M \longrightarrow N$ of oriented closed connected manifolds of the same dimension the estimate

$$|\deg f| \cdot \bigl\lvert [N]_{\mathbb{R}} \bigr\rvert \leq \bigl\lvert [M]_{\mathbb{R}} \bigr\rvert ;$$

hence, if $\lvert [N]_{\mathbb{R}} \rvert \neq 0$, then we obtain the restriction

$$|\deg f| \leq \frac{\bigl\lvert [M]_{\mathbb{R}} \bigr\rvert}{\bigl\lvert [N]_{\mathbb{R}} \bigr\rvert}$$

on the mapping degree. Such restrictions are particularly interesting when there are – at least for certain classes of (Riemannian) manifolds – estimates of $\lvert [\,\cdot\,]_{\mathbb{R}} \rvert$ in terms of the Riemannian volume or other geometric invariants.

For example, powerful degree theorems have been obtained by the use of simplicial volume and its variations [8, 16, Section 0.5, Section 1.2].

Conversely, in the following sections, we will translate knowledge about mapping degrees into constructions of functorial semi-norms with specific properties.

3. Representing homology classes by manifolds

As mentioned in the introduction, one of our main tools is to represent singular homology classes by manifolds. For the sake of completeness, we recall the following classical result:

Theorem 3.1. *Let X be a connected CW-complex, let $d \in \mathbb{N}$ and let $\alpha \in H_d(X;\mathbb{Q})$ be a singular homology class.*

(1) *Then there exists an $a \in \mathbb{Q} \setminus \{0\}$ and an oriented closed connected d-dimensional smooth manifold M together with a continuous map $f\colon M \longrightarrow X$ such that*

$$a \cdot H_d(f;\mathbb{Q})[M]_{\mathbb{Q}} = \alpha,$$

where $[M]_{\mathbb{Q}} \in H_d(M;\mathbb{Q})$ is the rational fundamental class of M.

(2) *If X is homotopy equivalent to a CW-complex with finite 2-skeleton and $d \geq 4$, then there exists an $a \in \mathbb{Q} \setminus \{0\}$ and an oriented closed connected d-dimensional manifold M together with a continuous map $f\colon M \longrightarrow X$ such that*

$$a \cdot H_d(f;\mathbb{Q})[M]_{\mathbb{Q}} = \alpha$$

and such that in addition $\pi_1(f)\colon \pi_1(M) \longrightarrow \pi_1(X)$ is an isomorphism.

Proof. The first part is a classical result by Thom [23].

For the second statement, we apply surgery theory as in [14]. Using the notation of *loc. cit.*, let $B := X \times \mathrm{BSO}$ where BSO is the classifying space of the stable special orthogonal group and let $B \longrightarrow \mathrm{BO}$ be the fibration given by projection to BSO and the canonical covering $\mathrm{BSO} \longrightarrow \mathrm{BO}$.

Given an oriented closed connected smooth manifold $\bar{\nu}\colon M \longrightarrow \mathrm{BSO}$ and a map $f\colon M \longrightarrow X$, we obtain the B-manifold $f \times \bar{\nu}\colon M \longrightarrow X \times \mathrm{BSO} = B$. Hence, there is an oriented bordism $F\colon W \longrightarrow X$ over X from $f\colon M \longrightarrow X$ to a map $g\colon N \longrightarrow X$ such that g is a 2-equivalence [14, Proposition 4]; in particular, g induces an isomorphism on fundamental groups. A straightforward computation in singular homology shows that

$$H_*(g;\mathbb{Q})[N]_\mathbb{Q} = H_*(f;\mathbb{Q})[M]_\mathbb{Q} - H_*(F;\mathbb{Q})[\partial W]_\mathbb{Q} = H_*(f;\mathbb{Q})[M]_\mathbb{Q};$$

choosing f as provided by part (1) finishes the proof. □

We next extend Theorem 3.1 to general path-connected spaces and to homology classes in $H_*(\,\cdot\,;\mathbb{R})$ which lie in the image of the change of coefficients homomorphism $H_*(\,\cdot\,;\mathbb{Q}) \longrightarrow H_*(\,\cdot\,;\mathbb{R})$. Such classes are called *rational*, and by the universal coefficients theorem, every class in $H_*(\,\cdot\,;\mathbb{R})$ is a finite $\mathbb{R}$-linear combination of rational classes.

Corollary 3.2. *Let X be a path-connected topological space, let $d \in \mathbb{N}$, and let $\alpha \in H_d(X;\mathbb{R})$ be rational.*

(1) *Then there exists an $a \in \mathbb{Q} \setminus \{0\}$ and an oriented closed connected smooth d-manifold M together with a continuous map $f\colon M \longrightarrow X$ such that $a \cdot H_d(f;\mathbb{R})[M]_\mathbb{R} = \alpha$, where $[M]_\mathbb{R} \in H_d(M;\mathbb{R})$ is the real fundamental class of M.*
(2) *If X is simply connected and $d \geq 4$, then there is an $a \in \mathbb{Q} \setminus \{0\}$, an oriented closed simply connected smooth d-manifold M, and a continuous map $f\colon M \longrightarrow X$ with $a \cdot H_d(f;\mathbb{R})[M]_\mathbb{R} = \alpha$.*

Proof. Out of the combinatorial data of a singular cycle in $C_d(X;\mathbb{R})$ representing α we can construct a connected finite CW-complex X', a rational homology class $\alpha' \in H_d(X';\mathbb{R})$ and a continuous map $f'\colon X' \longrightarrow X$ such that $H_d(f';\mathbb{R})(\alpha') = \alpha$; if X is simply connected, then we can also assume that X' is simply connected. Now the claim easily follows from the universal coefficient theorem and the previous theorem. □

4. Generating functorial semi-norms via special spaces

Every functorial semi-norm on singular homology induces by restriction a functorial semi-norm on the top homology of oriented closed connected manifolds. Conversely, examples of functorial semi-norms on singular homology can be generated by extending functorial semi-norms on the top homology of oriented closed connected manifolds (of a given dimension):

Definition 4.1 (Associated semi-norm)**.** Let $d \in \mathbb{N}$, let Mfd_d denote the class of all oriented closed connected d-manifolds, and let $S \subset \mathrm{Mfd}_d$ be a subclass.

– A *functorial semi-norm on fundamental classes of oriented closed connected d-manifolds in S*, or briefly a *functorial S-semi-norm*, is a map $v\colon S \longrightarrow [0,\infty]$ such that

$$|\deg f| \cdot v(N) \leq v(M)$$

holds for all continuous maps $f\colon M \longrightarrow N$ with $N, M \in S$.

If $S = \mathrm{Mfd}_d$, then we call such a v a *functorial semi-norm on fundamental classes of oriented closed connected d-manifolds*, briefly a *functorial* Mfd_d*-semi-norm*.

– Let v be an S-functorial semi-norm. The *associated semi-norm* $|\cdot|$ on singular homology in degree d is defined as follows: For a topological space X and a homology class $\alpha \in H_d(X;\mathbb{R})$ we set

$$\begin{aligned} |\alpha| := \inf\Biggl\{ \sum_{j=1}^k |a_j| \cdot v(M_j) \;\Biggm|\; & k \in \mathbb{N},\ a_1,\dots,a_k \in \mathbb{R} \setminus \{0\}, \\ & M_1,\dots,M_k \in S, \\ & f_1\colon M_1 \to X,\dots,f_k\colon M_k \to X \text{ continuous} \\ & \text{with } \sum_{j=1}^k a_j \cdot H_d(f_j;\mathbb{R})[M_j]_{\mathbb{R}} = \alpha \Biggr\}; \end{aligned}$$

we use the conventions $r \cdot \infty := \infty$ for all $r \in \mathbb{R}_{>0}$ and $\inf \emptyset := \infty$.

Theorem 4.2 (Generating functorial semi-norms). *Let $d \in \mathbb{N}$, let $S \subset \mathrm{Mfd}_d$ be a subclass, and let $|\cdot|$ be the semi-norm associated with a functorial semi-norm $v\colon S \longrightarrow [0,\infty]$ on fundamental classes of oriented closed connected d-manifolds in S (see Definition 4.1).*

(1) *Then $|\cdot|$ is a functorial semi-norm on singular homology in degree d, and for all oriented closed connected d-manifolds M in S we have*

$$\bigl|[M]_{\mathbb{R}}\bigr| = v(M).$$

(2) *If $S = \mathrm{Mfd}_d$ and v is finite, then so is $|\cdot|$.*

(3) *The associated semi-norm $|\cdot|$ is maximal in the following sense: If $|\cdot|'$ is a functorial semi-norm on singular homology in degree d that extends v, then $|\cdot|' \leq |\cdot|$.*

Proof. A straightforward computation shows that $|\cdot|$ as defined in Definition 4.1 is indeed a functorial semi-norm in degree d. If M is an oriented closed connected d-manifold in S, then representing $[M]_{\mathbb{R}}$ by $\mathrm{id}_M\colon M \longrightarrow M$ shows that $|[M]_{\mathbb{R}}| \leq v(M)$. On the other hand, $v(M) \leq |[M]_{\mathbb{R}}|$ as we now show. Let

$$[M]_{\mathbb{R}} = \sum_{j=1}^k a_j \cdot H_d(f_j;\mathbb{R})[M_j]_{\mathbb{R}} = \sum_{j=1}^k a_j \cdot \deg f_j \cdot [M]_{\mathbb{R}}$$

be a representation of $[M]_{\mathbb{R}}$ as in Definition 4.1; then $1 = \sum_{j=1}^k a_j \cdot \deg f_j$ and hence

$$v(M) \leq \sum_{j=1}^k |a_j| \cdot |\deg f_j| \cdot v(M) \leq \sum_{j=1}^k |a_j| \cdot v(M_j)$$

by functoriality of v on S. This proves the first part.

The second part follows from the fact that every real singular homology class of a path-connected space is an $\mathbb{R}$-linear combination of rational classes, which can – up to a non-zero factor – be represented by oriented closed connected manifolds (Corollary 3.2).

The last part follows directly from the construction of $\boldsymbol{|}\cdot\boldsymbol{|}$, the triangle inequality, and the definition of functoriality. □

For example, the ℓ^1-semi-norm can be viewed as the functorial semi-norm generated by simplicial volume:

Proposition 4.3. *Let $d \in \mathbb{N} \setminus \{3\}$. Then on the category of connected finite CW-complexes the functorial semi-norm on singular homology in degree d associated with the simplicial volume in dimension d coincides with the ℓ^1-semi-norm in degree d.*

Proof. Clearly, the statement holds in degree 0. In degree 1 the claim follows directly from the Hurewicz theorem.

In degree 2, one has to understand the simplicial volume of surfaces and how singular homology classes in degree 2 can be represented by surfaces: If S is an oriented closed connected surface of genus $g \geq 1$, then [8, 5, p. 9, Proposition C.4.7]

$$\|S\| = 4 \cdot g - 4 = 2 \cdot |\chi(S)|;$$

combining this fact with Proposition 2.4 proves the claim in degree 2.

Suppose now that the degree d is at least 4. In view of Theorem 4.2 (3), the functorial semi-norm $\boldsymbol{|}\cdot\boldsymbol{|}$ associated with the simplicial volume satisfies $\boldsymbol{|}\cdot\boldsymbol{|} \geq \|\cdot\|_1$; thus, it suffices to prove the reverse inequality.

Let X be a connected finite CW-complex, and let $\alpha \in H_d(X;\mathbb{R})$. We can write $\alpha = \sum_{j=1}^k a_j \cdot \alpha_j$, where $\alpha_1, \dots, \alpha_k \in H_d(X;\mathbb{R})$ are rational and $a_1, \dots, a_n \in \mathbb{R}$. For $n \in \mathbb{N}$ we let $\alpha^{(n)} := \sum_{j=1}^k a_j^{(n)} \cdot \alpha_j$, where $(a_j^{(n)})_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Q}$ that approximates a_j; by construction, the $\alpha^{(n)}$ are rational and the triangle inequality shows that

$$\lim_{n\to\infty} \boldsymbol{|}\alpha^{(n)} - \alpha\boldsymbol{|} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\alpha^{(n)} - \alpha\|_1 = 0.$$

Therefore, it suffices to prove $\boldsymbol{|}\cdot\boldsymbol{|} \leq \|\cdot\|_1$ for rational classes in $H_d(X;\mathbb{R})$.

If α is rational, then by Theorem 3.1, there is an $a \in \mathbb{R} \setminus \{0\}$, and a continuous map $f\colon M \longrightarrow X$ from some oriented closed connected d-manifold M such that

$$a \cdot H_d(f;\mathbb{R})[M]_{\mathbb{R}} = \alpha$$

and such that in addition $\pi_1(f)\colon \pi_1(M) \longrightarrow \pi_1(X)$ is an isomorphism. Applying the mapping theorem in bounded cohomology [8, 12, Section 3.1, Theorem 4.3] (combined with the duality principle for the ℓ^1-semi-norm [8, Corollary on p. 17]) shows that $H_d(f;\mathbb{R})\colon H_d(M;\mathbb{R}) \longrightarrow H_d(X;\mathbb{R})$ is isometric with respect to the ℓ^1-semi-norm. In particular,

$$\|\alpha\|_1 = |a| \cdot \big\|[M]_{\mathbb{R}}\big\|_1 = |a| \cdot \|M\| \geq \boldsymbol{|}\alpha\boldsymbol{|}. \qquad \square$$

The surface semi-norm is also a semi-norm defined as in Defintion 4.1:

Proposition 4.4. *Let $d \in \mathbb{N}$, let $S \subset \mathrm{Mfd}_{2\cdot d}$ be the subclass of products of d oriented closed connected surfaces, and let*

$$\begin{aligned} v\colon S &\longrightarrow [0,\infty] \\ M &\longmapsto |\chi(M)|. \end{aligned}$$

Then v is a functorial semi-norm on fundamental classes of oriented closed connected manifolds in S, and the functorial semi-norm on $H_{2\cdot d}(\,\cdot\,;\mathbb{R})$ associated with v is the surface semi-norm of Example 2.3.

Proof. That v indeed is functorial can, for example, be seen via the simplicial volume, the proportionality principle for simplicial volume, and the multiplicativity of the Euler characteristic [11, 15, p. 303, Corollary 6.5].

That the semi-norm associated with v and the surface semi-norm coincide follows directly from the definitions. □

Proposition 4.5. *The surface semi-norm $\lVert \cdot \rVert_S$ vanishes on all singular homology classes of simply connected spaces of degree 2 or 4.*

Proof. Let X be a simply connected topological space, and let $\alpha \in H_*(X;\mathbb{R})$ be a homology class of degree 2 or 4.

If α is of degree 2, then – because X is simply connected – we have an isomorphism $H_2(X;\mathbb{Z}) \cong \pi_2(X)$. Hence every integral homology class in degree 2 is represented by a map from the sphere S^2. Using the universal coefficient theorem and the fact that S^2 admits self-maps of arbitrarily large degree, it follows that the surface semi-norm vanishes on $H_2(X;\mathbb{R})$.

Let α now be of degree 4. In view of the triangle inequality, we can assume without loss of generality that α is rational. Then by Corollary 3.2 we can represent α as

$$a \cdot H_d(f;\mathbb{R})[M]_{\mathbb{R}} = \alpha,$$

where M is an oriented closed simply connected 4-manifold, $f\colon M \longrightarrow X$ is a continuous map, and $a \in \mathbb{R} \setminus \{0\}$. Moreover, the simply connected 4-manifold M is dominated by a product $S^1 \times S^1 \times S$, where S is a suitable oriented closed connected surface [13, Proposition 7.1]. Because $\chi(S^1 \times S^1 \times S) = 0$ it follows that $\lVert \alpha \rVert_S = 0$. □

Similarly to the definition of the surface semi-norm, we can also take hyperbolic manifolds as building blocks of a functorial semi-norm:

Example 4.6 (The hyperbolic semi-norm). Let $d \in \mathbb{N}$, let $H \subset \mathrm{Mfd}_d$ be the subclass of all oriented closed connected smooth d-manifolds that admit a hyperbolic Riemannian metric. Then

$$\begin{aligned} v\colon H &\longrightarrow [0,\infty] \\ M &\longmapsto \mathrm{vol}(M) \end{aligned}$$

is well-defined and functorial (because the volume of hyperbolic manifolds can be expressed in terms of the simplicial volume [8, 24, Section 0.3, Theorem 6.2] and because the simplicial volume is functorial).

We point out that it is still an open problem whether every manifold can be dominated by a hyperbolic manifold [13, Conjecture 7.2]; so it is not known whether the functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$ associated with v is finite.

Remark 4.7 (Generating functorial semi-norms via Poincaré spaces). Recall that a $\mathbb{Q}$-*Poincaré space of formal dimension* d is a connected CW-complex X together with a homology class $[X] \in H_d(X;\mathbb{Q})$, the *fundamental class*, such that

$$\cdot \cap [X] \colon H^*(X;\mathbb{Q}) \longrightarrow H_{*-d}(X;\mathbb{Q})$$

is an isomorphism. In particular, one can introduce the notion of mapping degree for continuous maps between $\mathbb{Q}$-Poincaré spaces of the same formal dimension.

Similarly to Definition 4.1 and Theorem 4.2, any functorial semi-norm on the fundamental classes of $\mathbb{Q}$-Poincaré complexes of a given dimension gives rise to an associated functorial semi-norm on singular homology in the given degree.

5. Functorial semi-norms (not) induced from the singular chain complex

One source of functorial semi-norms on singular homology is the class of functorial semi-norms on the singular chain complex: Let $d \in \mathbb{N}$. A *functorial semi-norm on the singular chain complex in degree d* consists of a choice of a semi-norm $|\cdot|$ on $C_d(X;\mathbb{R})$ for every topological space X such that the following "functoriality" holds: for all continuous maps $f \colon X \longrightarrow Y$ between topological spaces and all $c \in C_d(X;\mathbb{R})$ we have

$$\bigl|C_d(f;\mathbb{R})(c)\bigr| \leq |c| .$$

Such a functorial semi-norm on the singular chain complex is *finite* if all the semi-norms involved are finite semi-norms. For example, the ℓ^1-norm on the chain level (Example 2.2) is a finite functorial semi-norm on the singular chain complex.

(Non-)Example 5.1 (ℓ^p-Semi-norms). Let $d \in \mathbb{N}$, let $p \in (1,\infty]$, and let $|\cdot|_p$ be the p-norm on $C_d(\,\cdot\,;\mathbb{R})$ with respect to the (unordered) basis given by the set of all singular d-simplices. Then $|\cdot|_p$ is *not* a functorial semi-norm on the singular chain complex in degree d:

We consider $X := \{x,y\}$ with the discrete topology and $f \colon X \longrightarrow X$ mapping both points to x. Let $c := \sigma_x + \sigma_y \in C_d(X;\mathbb{R})$, where σ_x and σ_y are the constant singular d-simplices mapping to x and y respectively. Then

$$\bigl|C_d(f;\mathbb{R})(c)\bigr|_\infty = |2\cdot\sigma_x|_\infty = 2 > 1 = |\sigma_x + \sigma_y|_\infty = |c|_\infty,$$

and for $p \in (1,\infty)$ we obtain

$$\bigl|C_d(f;\mathbb{R})(c)\bigr|_p = |2\cdot\sigma_x|_p = 2 > \sqrt[p]{1^p + 1^p} = |\sigma_x + \sigma_y|_p = |c|_p$$

Hence $|\cdot|_p$ is *not* functorial.

Clearly, any [finite] functorial semi-norm on the singular chain complex in degree d induces a [finite] functorial semi-norm on singular homology in degree d by taking the infimum of the semi-norms of cycles representing a given class. Notice that being induced from a finite functorial semi-norm on the singular chain complex is a rather strong condition:

Proposition 5.2. *Let $d \in \mathbb{N}$ and let $\boldsymbol{|}\cdot\boldsymbol{|}$ be a finite functorial semi-norm on the singular chain complex in degree d. Then*

$$\boldsymbol{|}\cdot\boldsymbol{|} \;\leq\; \boldsymbol{|}\mathrm{id}_{\Delta^d}\boldsymbol{|} \cdot \|\cdot\|_1.$$

Proof. Let $c = \sum_{j=0}^{k} a_j \cdot \sigma_j \in C_d(X;\mathbb{R})$ be a singular chain (in reduced form). Viewing $\mathrm{id}_{\Delta^d} \colon \Delta^d \longrightarrow \Delta^d$ as a singular d-simplex on Δ^d, functoriality of $\boldsymbol{|}\cdot\boldsymbol{|}$ yields

$$\boldsymbol{|}c\boldsymbol{|} \;\leq \sum_{j=0}^{k} |a_j| \cdot \;\boldsymbol{|}\sigma_j \circ \mathrm{id}_{\Delta^d}\boldsymbol{|} \;\leq \sum_{j=0}^{k} |a_j| \cdot \;\boldsymbol{|}\mathrm{id}_{\Delta^d}\boldsymbol{|} \;=\; \boldsymbol{|}\mathrm{id}_{\Delta^d}\boldsymbol{|} \cdot \|c\|_1,$$

as desired. □

Corollary 5.3. *In particular, because the ℓ^1-semi-norm is trivial on simply connected spaces* [8, 12, Section 3.1, Theorem 2.4]*, every functorial semi-norm on singular homology induced from a* finite *functorial semi-norm on the singular chain complex is trivial on simply connected spaces.*

Concerning the converse question "Which [finite] functorial semi-norms on singular homology are induced from [finite] functorial semi-norms on the singular chain complex?", we prove in the following:

- Every functorial semi-norm on singular homology is induced from some (in general infinite) functorial semi-norm on the singular chain complex (Proposition 5.4);
- There exist finite functorial semi-norms on singular homology that are not induced from a *finite* functorial semi-norm on the singular chain complex (Theorem 5.7).

  So, Corollary 5.3 is not strong enough to answer Gromov's question (Question 1.1(2)) in the positive for *all* finite functorial semi-norms.

Proposition 5.4. *Let $d \in \mathbb{N}$, and let $\boldsymbol{|}\cdot\boldsymbol{|}$ be a functorial semi-norm on singular homology in degree d. Then there is a functorial semi-norm $|\cdot|$ on the singular chain complex in degree d inducing $\boldsymbol{|}\cdot\boldsymbol{|}$: i.e., for all topological spaces X and all $\alpha \in H_d(X;\mathbb{R})$ we have*

$$\boldsymbol{|}\alpha\boldsymbol{|} = \inf\bigl\{|c| \bigm| c \in C_d(X;\mathbb{R}) \text{ is a cycle representing } \alpha\bigr\}.$$

Proof. Let X be a topological space. We denote by $i \colon Z_d(X;\mathbb{R}) \longrightarrow C_d(X;\mathbb{R})$ and $p \colon Z_d(X;\mathbb{R}) \longrightarrow H_d(X;\mathbb{R})$ the inclusion of the d-cycles and the projection onto the d-th homology group respectively. We define a semi-norm $|\cdot|$ on $C_d(X;\mathbb{R})$ by setting

$$|\cdot| := i_* p^* \,\boldsymbol{|}\cdot\boldsymbol{|}\,,$$

where i_* and p^* are defined as follows:

(1) *Construction of $p^*|\cdot|$:* Let $p \colon V \longrightarrow U$ be a surjective homomorphism of $\mathbb{R}$-vector spaces, and let $|\cdot|$ be a semi-norm on U. Then

$$\begin{aligned} p^*|\cdot| \colon V &\longrightarrow [0,\infty] \\ x &\longmapsto |p(x)| \end{aligned}$$

is a semi-norm on V (this is a straightforward calculation).

(2) *Construction of* $i_*|\cdot|$: Let $i\colon U \longrightarrow V$ be the inclusion of a subspace of an $\mathbb{R}$-vector space, and let $|\cdot|$ be a semi-norm on U. Then

$$\begin{aligned} i_*|\cdot|\colon V &\longrightarrow [0,\infty] \\ x &\longmapsto \begin{cases} |x| & \text{if } x \in U, \\ \infty & \text{if } x \in V \setminus U \end{cases} \end{aligned}$$

is a semi-norm on V; clearly, $i_*|0| = |0| = 0$, and $i_*|\cdot|$ is compatible with scalar multiplication. Moreover, the triangle inequality is satisfied: Let $x, y \in V$. If $x \in V \setminus U$ or $y \in V \setminus U$, then $i_*|x| = \infty$ or $i_*|y| = \infty$, so that the triangle inequality is trivially satisfied. The only remaining case is that $x, y \in U$, and in this case the triangle inequality is satisfied, because $|\cdot|$ is a semi-norm on U.

Note that if $U \neq V$, then $i_*|\cdot|$ is infinite.

Why is $|\cdot| = i_* p^* \,|\!|\cdot|\!|$ functorial? Let $f\colon X \longrightarrow Y$ be a continuous map and let $c \in C_d(X;\mathbb{R})$. If c is not a cycle, then $|c| = \infty$, and so $|C_d(f;\mathbb{R})(c)| \leq |c|$. In case c is a cycle, then $C_d(f;\mathbb{R})(c)$ is a cycle as well and thus

$$\bigl|C_d(f;\mathbb{R})(c)\bigr| = \bigl|\!\bigl|[C_d(f;\mathbb{R})(c)]\bigr|\!\bigr| = \bigl|\!\bigl|H_d(f;\mathbb{R})[c]\bigr|\!\bigr| \leq |\!|[c]|\!| = |c|$$

because $|\!|\cdot|\!|$ is functorial.

Moreover, $|\cdot|$ induces $|\!|\cdot|\!|$ on homology because for all cycles c we have $|\!|[c]|\!| = |c|$ by construction of $|\cdot|$. □

However, even if the given functorial semi-norm on singular homology is finite, the corresponding functorial semi-norm on the singular chain complex provided in the proof of Proposition 5.4 is *not* finite. This is not merely an artefact of this construction: in the following we give an example of a finite functorial semi-norm on singular homology that grows too fast (compared to the ℓ^1-semi-norm) to be induced from a finite functorial semi-norm on the singular chain complex.

Definition 5.5 (Degree monotonic map)**.** A function $\varphi\colon \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ that is monotonically growing is called *degree monotonic* if for all $x \in \mathbb{R}_{\geq 0}$ and all $d \in \mathbb{N}$ we have

$$\varphi(d \cdot x) \geq d \cdot \varphi(x).$$

Proposition 5.6. *Let $d \in \mathbb{N}$ and let $v\colon \operatorname{Mfd}_d \longrightarrow \mathbb{R}_{\geq 0}$ be a finite functorial semi-norm on fundamental classes of oriented closed connected d-manifolds. If $\varphi\colon \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a degree monotonic map, then the composition*

$$\varphi \circ v\colon \operatorname{Mfd}_d \longrightarrow \mathbb{R}_{\geq 0}$$

is a finite functorial semi-norm on fundamental classes of oriented closed connected d-manifolds.

Proof. For all continuous maps $f\colon M \longrightarrow N$ between oriented closed connected d-manifolds, we have $v(M) \geq |\deg f| \cdot v(N)$, and thus

$$\varphi \circ v(M) \geq \varphi\bigl(|\deg f| \cdot v(N)\bigr) \geq |\deg f| \cdot \varphi \circ v(N)$$

by the degree monotonicity of φ. □

Theorem 5.7. *There are finite functorial semi-norms on singular homology that are not induced from a finite functorial semi-norm on the singular chain complex.*

Proof. Let $\varphi\colon \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ be a degree monotonic map that grows faster than linearly in the sense that $\lim_{x\to\infty} \varphi(x)/x = \infty$; for instance, for every $a \in \mathbb{R}_{>1}$ the map

$$\begin{aligned} \mathbb{R}_{\geq 0} &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto x^a \end{aligned}$$

has this property. Moreover, let $d \in \mathbb{N}_{\geq 2}$.

We now consider the functorial semi-norm $|\cdot|$ on singular homology in degree d associated with the finite functorial semi-norm on fundamental classes of oriented closed connected d-manifolds given by composing φ with the simplicial volume (Proposition 5.6 and Theorem 4.2); notice that $|\cdot|$ is finite.

Assume for a contradiction that $|\cdot|$ were induced from a finite functorial semi-norm. Then in view of Proposition 5.2 we would have

$$(*) \qquad |\cdot| \leq |\mathrm{id}_{\Delta^d}| \cdot \|\cdot\|_1.$$

However, we now show that $|\cdot|$ "grows too fast" to be able to satisfy this estimate. To see this consider the properties of hyperbolic manifolds more closely: Let M be an oriented closed connected hyperbolic d-manifold. Then the fundamental group $\pi_1(M)$ of M is residually finite [20, p. 542]; so for any $k \in \mathbb{N}$ there is a subgroup $\Gamma_k \subset \pi_1(M)$ satisfying

$$k \leq \bigl[\pi_1(M) : \Gamma_k\bigr] < \infty.$$

For $k \in \mathbb{N}$ we let $p_k\colon M_k \longrightarrow M$ denote the covering associated with the inclusion $\Gamma_k \subset \pi_1(M)$; hence, M_k also is an oriented closed connected (hyperbolic) d-manifold and

$$|\deg p_k| = \bigl[\pi_1(M) : \Gamma_k\bigr] \geq k.$$

Because M is hyperbolic, the simplicial volume $\|M\|$ is non-zero [8, 24, Section 0.3, Theorem 6.2]; thus, $\|M_k\| \geq k \cdot \|M\|$ tends to ∞ for $k \to \infty$. By definition, φ grows faster than linearly and so

$$\frac{\bigl|[M_k]_{\mathbb{R}}\bigr|}{\bigl\|[M_k]_{\mathbb{R}}\bigr\|_1} = \frac{\varphi(\|M_k\|)}{\|M_k\|}$$

tends to ∞ for $k \to \infty$, contradicting the estimate in Equation $(*)$. Therefore, the finite functorial semi-norm $|\cdot|$ on singular homology is not induced from a finite functorial semi-norm on the singular chain complex. $\square$

Question 5.8. *In light of the example constructed in the proof of Theorem 5.7, it is natural to ask for a reasonable notion of equivalence of functorial semi-norms on singular homology or for a notion of domination of one functorial semi-norm by another. Is the ℓ^1-semi-norm on singular homology "maximal" among finite functorial semi-norms on singular homology with respect to such a notion? (This should also be compared with Proposition 7.6.)*

6. (IN)FLEXIBLE MANIFOLDS

The constructions of interesting functorial semi-norms in Section 7.1 below require as input simply connected manifolds that are inflexible; recall that an oriented closed connected manifold M is *inflexible* if it admits only self-maps of degree 0, 1 or -1, i.e., $\deg(M,M) \subset \{-1,0,1\}$.

Remark 6.1. Looking at iterated compositions shows that an oriented closed connected manifold M is flexible if and only if $|\deg(M,M)| = \infty$. Conversely, the manifold M is inflexible if and only if $\deg(M,M)$ is finite.

Remark 6.2. If a manifold is flexible, then – by functoriality – its simplicial volume is zero. In particular, oriented closed connected hyperbolic manifolds are inflexible, as they have non-zero simplicial volume. However, for simply connected manifolds the simplicial volume is zero and hence the simplicial volume cannot serve as an obstruction to flexibilty in this case.

In this section we show how rational homotopy theory and surgery allow one to construct examples of simply connected inflexible manifolds, building upon examples of Arkowitz and Lupton [2] (Sections 6.1 and 6.2). We briefly discuss strongly inflexible manifolds in Section 6.3. Finally, in Section 6.4, we discuss the class of simply connected flexible manifolds from the viewpoint of rational homotopy theory. To make this section more readable we have moved most of the calculations with differential graded algebras and the proof of inheritance properties of simply connected inflexible manifolds to the appendices Section 8 and 9.

6.1. **(In)Flexibility and rational homotopy theory.** We start by giving an overview of the construction of simply connected inflexible manifolds and introducing key notations and definitions.

Rational homotopy theory provides the rationalisation functor $\cdot_{\mathbb{Q}}$ on the category of simply connected spaces and an equivalence of categories between the category of simply connected rational spaces and the category of certain differential graded algebras, the so-called minimal models. For the basic definitions in rational homotopy theory, we refer to the book by Félix, Halperin and Thomas [7].

More concretely, if M is an oriented closed simply connected manifold, then the associated minimal model A_M is a differential graded algebra over $\mathbb{Q}$ whose cohomology coincides with the rational cohomology of M; in particular, A_M has a cohomological fundamental class $[A_M]$, namely the cohomology class of $H^*(A_M) \cong H^*(M;\mathbb{Q})$ dual to the fundamental class $[M]_{\mathbb{Q}}$ of M.

Any self-map $f\colon M \longrightarrow M$ induces a corresponding dga endomorphism $A_f\colon A_M \longrightarrow A_M$; using the cohomological fundamental class $[A_M]$ of A_M we can associate a mapping degree to A_f, and this mapping degree coincides with $\deg(f)$. In particular, if A_M is "inflexible", as defined in Definition 6.5 below, then so is M.

Hence, it suffices to find differential graded algebras that are minimal models of simply connected manifolds and whose cohomological fundamental class is inflexible; notice that the latter condition is algebraic by defintion and moreover that Theorem 6.11 below entails that this is also true of the former condition.

We now give a precise definition of inflexibility and duality in the world of differential graded algebras:

Definition 6.3 ((In)flexible (co)homology classes)**.**

- A homology class $\alpha \in H_*(X;\mathbb{Q})$ of a topological space X is *flexible* if there is a continuous map $f\colon X \longrightarrow X$ such that
$$H_*(f;\mathbb{Q})(\alpha) = d \cdot \alpha$$
for some $d \in \mathbb{Q} \setminus \{-1,0,1\}$. A homology class is called *inflexible* if it is not flexible.
  (In particular, an oriented closed connected manifold is inflexible if and only if its fundamental class is inflexible).
- A cohomology class $\alpha \in H^*(A)$ of a differential graded algebra A is *flexible* if there is a dga endomorphism $f\colon A \longrightarrow A$ such that
$$H^*(f)(\alpha) = d \cdot \alpha$$
for some $d \in \mathbb{Q} \setminus \{-1,0,1\}$. A cohomology class is *inflexible* if it is not flexible.

Definition 6.4 (Poincaré differential graded algebra)**.** Let $n \in \mathbb{N}$. A *Poincaré differential graded algebra of formal dimension n* is a simply connected differential graded algebra A together with a cohomology class $[A] \in H^n(A)$, the *fundamental class*, satisfying the following conditions:

(1) For all $j \in \mathbb{N}_{>n}$ we have $H^j(A) = 0$.
(2) The map
$$\begin{aligned} \mathbb{Q} &\longrightarrow H^n(A) \\ a &\longmapsto a \cdot [A] \end{aligned}$$
is an isomorphism.
(3) For all $j \in \{0,\dots,n\}$, the pairing $H^j(A) \times H^{n-j}(A) \longrightarrow H^n(A) \cong \mathbb{Q}$ (where we use the isomorphism $H^n(A) \cong \mathbb{Q}$ of the previous item) given by multiplication identifies $H^j(A)$ with $\operatorname{Hom}_{\mathbb{Q}}(H^{n-j}(A),\mathbb{Q})$.

Definition 6.5 (Inflexible Poincaré algebra/space)**.**

- A Poincare differential graded algebra $(A,[A])$ is *inflexible*, if its fundamental class $[A]$ is inflexible in the sense of Definition 6.3.
- A $\mathbb{Q}$-Poincaré space $(X,[X])$ (see Remark 4.7 for a definition) is *inflexible*, if its fundamental class $[X]$ is inflexible in the sense of Definition 6.3.

6.2. **Simply connected inflexible manifolds.** Arkowitz and Lupton gave examples of differential graded algebras that admit only finitely many homotopy classes of dga endomorphisms [2, Examples 5.1 and 5.2]. Moreover, they showed how to prove that these differential graded algebras are minimal models of simply connected closed manifolds. In particular, these simply connected manifolds are inflexible.

In Section 8 we review their construction, and give two more examples of differential graded algebras with inflexible fundamental class:

Theorem 6.6. *There are inflexible Poincaré differential graded algebras $(A_1,[A_1])$, $(A_2,[A_2])$, $(A_3,[A_3])$, $(A_4,[A_4])$ of formal dimensions* 64, 108, 208, *and* 228 *respectively.*

Proof. This is proved in Section 8 (Corollary 8.7 and Proposition 8.10), where also the choice of fundamental class is specified (Proposition 8.6). □

In the following, we focus on the realisability of these Poincaré differential graded algebras by simply connected manifolds (for simplicity, we consider only the case of trivial total Pontryagin class):

Definition 6.7 (Realisability by manifolds)**.** Suppose $(A, [A])$ is a Poincaré differential graded algebra of formal dimension n. We then write $\mathcal{M}(A, [A])$ for the class of all oriented closed simply connected n-manifolds M that have trivial total Pontryagin class and that satisfy

$$(A_M, [A_M]) \cong (A, [A]).$$

Theorem 6.8 (Simply connected inflexible manifolds)**.** *For the above Poincaré dgas* $(A_1, [A_1]), \dots, (A_4, [A_4])$ *the classes* $\mathcal{M}(A_1, [A_1]), \dots, \mathcal{M}(A_4, [A_4])$ *are non-empty. In particular, there are oriented closed simply connected inflexible manifolds of dimension* 64, 108, 208, 228 *respectively.*

We now assemble the statements we need to prove Theorem 6.8. As first step, we show that the differential graded algebras $A_1, \dots, A_4$ are the corresponding dgas of rational $\mathbb{Q}$-Poincaré spaces:

Proposition 6.9 (Realisibility by $\mathbb{Q}$-Poincaré spaces)**.** *For the above Poincaré dgas* $(A_1, [A_1]), \dots, (A_4, [A_4])$ *there are corresponding simply connected rational* $\mathbb{Q}$*-Poincaré spaces* $(X_1, [X_1]), \dots, (X_4, [X_4])$ *respectively realising these dgas as their minimal models such that the cohomology classes corresponding to the fundamental classes* $[A_j]$ *are dual to the fundamental classes* $[X_j]$*; these spaces* $X_1, \dots, X_4$ *are unique up to rational homotopy equivalence, and they have formal dimension*

$$64,\ 108,\ 208,\ 228$$

respectively.

Proof. Because the dgas $A_1, \dots, A_4$ are Poincaré, the correspondence between rational spaces and minimal Sullivan algebras [7, Chapter 17] shows that up to rational homotopy equivalence there is a unique simply connected rational space that is a $\mathbb{Q}$-Poincaré space whose minimal model is A_1, A_2, A_3 or A_4 respectively, and whose fundamental class corresponds to the fundamental class of the respective dga.

Moreover, there is a formula expressing the formal dimension in terms of the degrees of the generators of an elliptic dga [7, Proposition 38.3]: The generators for our examples along with their degrees are given in Section 8.1 and the calculation boils down to the formal dimension

$$|y_1| + |y_2| + |y_3| + |z| - \big(|x_1| - 1\big) - \big(|x_2| - 1\big),$$

and hence to the formal dimensions 64, 108, 208, and 228 respectively. □

Corollary 6.10 (Inflexible $\mathbb{Q}$-Poincaré spaces)**.** *In particular, the simply connected rational* $\mathbb{Q}$*-Poincaré spaces* $(X_1, [X_1]), \dots, (X_4, [X_4])$ *from Proposition 6.9 are inflexible.*

Proof. Let $j \in \{1, \dots, 4\}$. *Assume* for a contradiction that X_j is flexible. Then there is a continuous map $f \colon X_j \longrightarrow X_j$ of degree $d \notin \{-1, 0, 1\}$. The map f induces a dga morphism $A_j \longrightarrow A_j$ of degree d, because A_j is a minimal model of X_j. However, this contradicts inflexibility of the dga A_j established in Proposition 8.10. □

It now remains to show that the rational $\mathbb{Q}$-Poincaré spaces of Corollary 6.10 can be realised by simply connected manifolds. To this end, we apply a foundational theorem of Barge [3] and Sullivan [22] (a special case is Theorem 6.11 below). This theorem gives necessary and sufficient conditions for a rational $\mathbb{Q}$-Poincaré space X to be realised by a manifold with prescribed rational Pontryagin classes; moreover the conditions are formulated using only the rational cohomology ring of X. Before stating the theorem we recall some basic terminology:

Let $\lambda \colon H \otimes H \longrightarrow \mathbb{Q}$ be a non-singular symmetric bilinear form over a finite dimensional $\mathbb{Q}$-vector space H. Recall that a *Lagrangian* for (H, λ) is a subspace $L \subset H$ such that $\lambda|_{L\times L} = 0$ and $2 \cdot \operatorname{rank}(L) = \operatorname{rank}(H)$; the pair (H, λ) is called *metabolic* if it admits a Lagrangian. The *Witt group of* $\mathbb{Q}$, denoted $W_0(\mathbb{Q})$, is the Grothendieck group of the monoid of isomorphism classes of non-singular symmetric bilinear forms on finite dimensional $\mathbb{Q}$-vector spaces under the operation of direct sum and modulo the subgroup generated by differences of metabolic forms [17, I § 7].

If $(X, [X])$ is a $\mathbb{Q}$-Poincaré space of formal dimension $4k$ then the cup-product followed by evaluation on $[X]$ defines a non-singular symmetric bilinear form $(H^{2k}(X;\mathbb{Q}), \lambda_{[X]})$. The *Witt index of* $(X, [X])$ is defined to be the equivalence class of this form in the Witt group of $\mathbb{Q}$:

$$\tau_{[X]} := \left[H^{2k}(X;\mathbb{Q}), \lambda_{[X]}\right] \in W_0(\mathbb{Q}).$$

Theorem 6.11 (Realising rational $\mathbb{Q}$-Poincaré spaces by manifolds [3, 22, Théorème 1, Theorem 13.2])**.** *Suppose that $(X, [X])$ is a rational $\mathbb{Q}$-Poincaré complex of formal dimension $4k$ and that $p_* \in H^{4*}(X;\mathbb{Q})$ is a cohomology class with $p_0 = 1 \in H^0(X;\mathbb{Q})$. Then there is an oriented closed simply connected manifold $(M, [M])$ with total Pontryagin class p_M and a rational equivalence $f \colon M \longrightarrow X$ with $H_{4k}(f;\mathbb{Q})([M]_\mathbb{Q}) = [X]$ and $H^{4k}(f;\mathbb{Q})(p_M) = p_*$ if and only if the following two conditions hold:*

1. *The Witt index $\tau_{[X]}$ of $(X, [X])$ lies in the image of the homomorphism $W_0(\mathbb{Z}) \longrightarrow W_0(\mathbb{Q})$.*
2. *There is an equality $\operatorname{sign}(X, [X]) = \langle L(p_*), [X]\rangle$ where $L(p_*)$ is the Hirzebruch L-class evaluated at p_*.*

Proposition 6.12 (Witt index)**.** *Let $j \in \{1, \dots, 4\}$, and let $(X_j, [X_j])$ be the corresponding $\mathbb{Q}$-Poincaré space of Proposition 6.9 (oriented by the choice of fundamental class in Proposition 8.6). Then $(X_j, [X_j])$ has trivial Witt index, i.e., $\tau_{[X_j]} = 0 \in W_0(\mathbb{Q})$. In particular, the signature $\operatorname{sign}(X_j)$ of $(X_j, [X_j])$ equals 0.*

Proof. The result follows by explicit computation. For example, the intersection form of $(A_1, [x_2^{16}])$ and hence $(X_1, [X_1])$ is computed in Proposition 8.8 where a basis for the middle cohomology is given. With respect to this matrix, the intersection matrix is an element of $\mathrm{GL}(4, \mathbb{Z})$ and has Lagrangian with basis $\{[x_2 w], [x_1^2 w]\}$. Similar calculations prove the proposition for A_2, A_3 and A_4. □

Proof of Theorem 6.8. Let $j \in \{1, \dots, 4\}$, and let $(X_j, [X_j])$ be the simply connected rational $\mathbb{Q}$-Poincaré space provided by Proposition 6.9. In view of Proposition 6.12, the Witt index $\tau_{[X_j]}$ lies in the image of the homomorphism $W_0(\mathbb{Z}) \longrightarrow W_0(\mathbb{Q})$; choosing $p := 1 \in H^0(X_j; \mathbb{Q}) \subset H^*(X_j; \mathbb{Q})$, we obtain

$$\operatorname{sign}(X_j, [X_j]) = 0 = \langle L(p), [X] \rangle.$$

Therefore, by Theorem 6.11, there exists an oriented closed simply connected manifold $(M, [M])$ rationally equivalent to $(X_j, [X_j])$ with trivial Pontryagin class; because $(A_j, [A_j])$ is the minimal model of $(X_j, [X_j])$, it follows that $M \in \mathcal{M}(A_j, [A_j])$.

In particular, this manifold M is inflexible (using the same arguments as in the proof of Corollary 6.10). □

Remark 6.13 (Scaling the fundamental class). The results of Theorem 6.8, Proposition 6.9, Corollary 6.10, and Proposition 6.12 all hold if the fundamental classes of the respective dgas/Poincaré complexes are scaled by any non-zero rational number. The key point is that if λ is a non-singular symmetric bilinear form on a finite dimensional $\mathbb{Q}$-vector space that is trivial in the Witt group $W_0(\mathbb{Q})$ and if $a \in \mathbb{Q} \setminus \{0\}$, then also $a \cdot \lambda$ is trivial in the Witt group (because any Lagrangian for λ also is a Lagrangian for $a \cdot \lambda$). Notice that scalars with different absolute values lead to different homotopy types of simply connected inflexible manifolds in the same rational homotopy type (Proposition 9.8).

Starting with the manifolds in $\mathcal{M}(A_1, [A_1]), \dots, \mathcal{M}(A_4, [A_4])$ we can construct many more simply connected inflexible manifolds; a detailed discussion of these results is deferred to Section 9.

6.3. **Strongly inflexible manifolds.** A manifold M is inflexible if and only if the set $\deg(M, M)$ is finite. More ambitiously we can ask that $\deg(N, M)$ is finite for any oriented manifold N of the same dimension as M. This leads to the notion of strongly inflexible manifolds:

Definition 6.14 (Strongly inflexible manifold)**.** We call an oriented closed connected d-dimensional manifold M *strongly inflexible* if for any oriented closed connected d-dimensional manifold N the set $\deg(N, M)$ is finite.

Clearly, any strongly inflexible manifold is also inflexible.

Example 6.15. The simplicial volume can be used to show that oriented closed connected hyperbolic manifolds M are strongly inflexible: If N is an oriented closed connected manifold of dimension $\dim M$, then

$$|\deg f| \leq \frac{\|N\|}{\|M\|} < \infty$$

for any map $f \colon N \longrightarrow M$; notice that $\|M\| > 0$ as M is hyperbolic.

Unfortunately, we do not know of any *simply connected* manifolds that are strongly inflexible. As in the case of inflexible manifolds, rational homotopy theory and the examples from Section 6.2 and Section 8 could be a good starting point for seeking strongly inflexible manifolds. However one sees that the necessary calculations, if they are possible, would be significantly more complicated than in the case of inflexible manifolds.

Question 6.16. *Is every "random" Poincaré differential graded algebra of high formal dimension (strongly) inflexible?*

A small piece of evidence supporting a positive answer to Question 6.16 is the bordism result in Proposition 9.12.

6.4. **Flexible spaces and manifolds.** Clearly, all spheres (of non-zero dimension) are flexible manifolds, and products of oriented closed connected manifolds with flexible ones are flexible again. Further examples of flexible manifolds and spaces can be obtained via rational homotopy theory:

Proposition 6.17 (Simply-connected manifolds of low dimension are flexible)**.** *Oriented closed simply connected formal manifolds are flexible. In particular, all oriented closed simply connected manifolds of dimension* 6 *or less are flexible.*

Proof. Formal oriented closed simply connected manifolds admit many self-maps of non-trivial degree [21] and so are flexible. Moreover, by a classical result in rational homotopy theory, all oriented closed simply connected manifolds of dimension at most 6 are formal [19, Proposition 4.6]; □

A natural generalisation of formality of minimal models is being pure:

Definition 6.18 (Pure)**.** A Sullivan algebra $(\bigwedge V, d)$ is *pure* if V is finite dimensional and

$$d|_{V^{\text{even}}} = 0 \qquad \text{and} \qquad d(V^{\text{odd}}) \subset \bigwedge V^{\text{even}};$$

here, V^{even} and V^{odd} denote the even and the odd part respectively of the graded vector space V.

Proposition 6.19 (Pure rational spaces are "almost flexible")**.** *Let X be a rational space whose minimal model is pure. Then every rational homology class of X in positive degree is a sum of flexible homology classes.*

Proof. Let $A = (\bigwedge V, d)$ be the minimal model of X. In view of the equivalence of categories between the category of minimal Sullivan dgas (and homotopy classes of dga morphisms) and the category of rational spaces (and homotopy classes of continuous maps) it suffices to show that every cohomology class in $H^*(\bigwedge V, d) \cong H_*(X; \mathbb{Q})$ in positive degree is a sum of flexible cohomology classes (as defined in Definition 6.3).

Let $f\colon \bigwedge V \longrightarrow \bigwedge V$ be the algebra morphism uniquely determined by the maps

$$\begin{aligned} V^{\text{even}} &\longrightarrow V \\ x &\longmapsto 2^{|x|} \cdot x, \\ V^{\text{odd}} &\longrightarrow V \\ y &\longmapsto 2^{|y|-1} \cdot y. \end{aligned}$$

Using the fact that $(\bigwedge V, d)$ is pure, a straightforward computation shows that f is compatible with d: On the even part, the differential vanishes, and so $f \circ d|_{V^{\text{even}}} = 0 = d \circ f|_{V^{\text{even}}}$. The differential of an odd element $y \in V^{\text{odd}}$

of V is a sum of products of even elements of V whose degrees sum up to $|y| - 1$, and so

$$f \circ d(y) = 2^{|y|-1} \cdot dy = d \circ f(y).$$

Because A is pure, there is an additional grading on A given by the word length in V^{odd}; more explicitly, $A = \bigoplus_{k \in \mathbb{N}} A_{[k]}$, where

$$A_{[k]} := \bigwedge V^{\text{even}} \otimes \bigwedge^k V^{\text{odd}}$$

for all $k \in \mathbb{N}$ [7, p. 435]; notice that the differential d is homoegeneous of degree -1 with respect to this grading and that $f(z) = 2^{|z|-k} \cdot z$ holds for all $k \in \mathbb{N}$ and all $z \in A_{[k]}$.

So the dga morphism f witnesses that every cohomology class in $H^*(A)$ of non-zero degree that can be represented by a cocycle in one of the subspaces $A_{[k]}$ is flexible. On the other hand, using the direct sum decomposition $A = \bigoplus_{k \in \mathbb{N}} A_{[k]}$ and the fact that d is homogeneous of degree -1 one can easily check that every cohomology class in $H^*(A)$ is a sum of cohomology classes represented by such cocycles. □

Flexibility as established in Proposition 6.17 and 6.19 provides a means to prove the vanishing of finite functorial semi-norms on certain classes (Corollary 7.7 and 7.8). Clearly, the same methods apply whenever the minimal models allow for an approriate grading or weight function. For simplicity, we restricted ourselves to the cases above.

7. FUNCTORIAL SEMI-NORMS ON SIMPLY CONNECTED SPACES

In the following we discuss Gromov's question whether all functorial semi-norms on singular homology are trivial on simply connected spaces (Question 1.1).

Here, a key rôle is played by simply connected inflexible manifolds. Recall that an oriented closed connected manifold M is *inflexible* if

$$\deg(M, M) \subset \{-1, 0, 1\}.$$

We start with a construction of a functorial semi-norm that is not trivial on all simply connected spaces (Section 7.1); on the other hand, we show in Section 7.2 that the finite case of Gromov's question can be answered affirmatively in all dimensions $d \leq 6$.

7.1. **Functorial semi-norms that are non-trivial on certain simply connected spaces.**

Using the construction from Section 4 and simply connected inflexible manifolds, we obtain a (possibly infinite) functorial semi-norm that is non-trivial on simply connected spaces:

Recall that an oriented closed connected manifold N is said to *dominate* an oriented closed connected manifold M of the same dimension if there exists a continuous map $N \longrightarrow M$ of non-zero degree.

Definition 7.1 (Domination Mfd_d-semi-norm associated with a d-manifold)**.** Let M be an oriented closed connected d-manifold. Then the *domination*

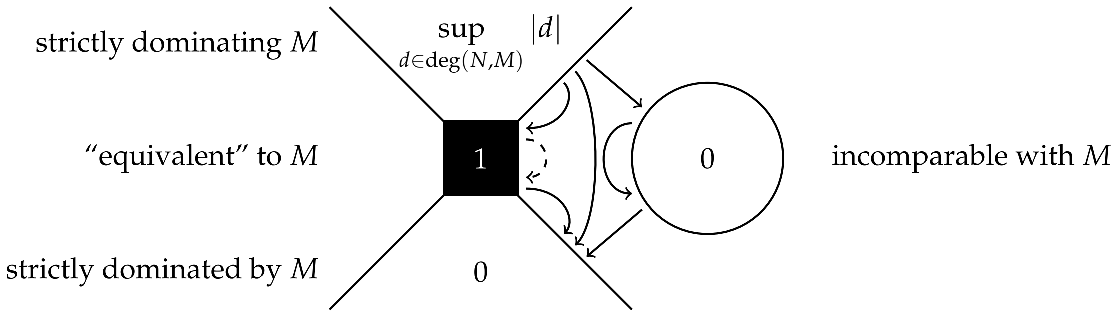

FIGURE 1. Schematic construction of v_M in the proof of Corollary 7.4; the arrows indicate where maps of non-zero degree can exist, the dashed arrow indicates that only maps of degree $-1, 0, 1$ can exist.

Mfd_d*-semi-norm* $v_M \colon \mathrm{Mfd}_d \longrightarrow [0,\infty]$ *associated with* M is defined by

$$\begin{aligned} v_M(N) &:= \sup\{|d| \mid d \in \deg(N,M)\} \\ &= \sup\{|\deg f| \mid f \colon N \longrightarrow M \text{ continuous}\} \in [0,\infty] \end{aligned}$$

for all $N \in \mathrm{Mfd}_d$.

Proposition 7.2. *If M is an oriented closed connected d-manifold, then the domination* Mfd_d*-semi-norm v_M is functorial.*

Proof. This follows from the definition of the domination semi-norm and multiplicativity of the mapping degree. □

Definition 7.3 (Domination semi-norm associated with a d-manifold)**.** Let M be an oriented closed connected d-manifold. Then the *domination semi-norm* $\lvert\cdot\rvert_M$ *on singular homology of degree d* is the semi-norm on singular homology in degree d associated with v_M (see Definition 4.1). By Proposition 7.2 and Theorem 4.2, $\lvert\cdot\rvert_M$ is a functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$.

Corollary 7.4. *If M is a simply connected closed inflexible manifold, then the domination semi-norm $\lvert\cdot\rvert_M$ is not zero or infinite on all simply connected spaces. Hence there are functorial semi-norms on singular homology that are not zero or infinite on all simply connected spaces.*

Proof. Let M be a simply connected closed inflexible manifold; such a manifold exists by Theorem 6.8 – we can even find such manifolds in infinitely many different dimensions (Corollary 9.7). By Theorem 4.2 (1) we have

$$\lvert [M]_{\mathbb{R}} \rvert_M = v_M(M) = 1 \notin \{0,\infty\}.$$

Here, $[M]_{\mathbb{R}}$ is of course the $\mathbb{R}$-fundamental class of the simply connected closed manifold M. We give a graphical description of the domination semi-norm associated to M in Figure 1. □

Remark 7.5. We do not know whether the functorial semi-norms constructed in Corollary 7.4 are finite. If M is a an oriented closed connected d-manifold then by construction the domination Mfd_d-semi-norm v_M is finite if and only if M is strongly inflexible. It then follows by Theorem 4.2 (2) and the definitions that the associated functorial semi-norm $|\cdot|_M$ is finite if and only if M is strongly inflexible. The existence of simply-connected strongly inflexible manifolds remains open at the time of writing.

As we do not know of any simply connected strongly inflexible manifold, Gromov's question (Question 1.1(2)) remains open for *finite* functorial semi-norms on singular homology.

7.2. **Partial results on finite functorial semi-norms on simply connected spaces.** In view of the Hurewicz theorem, all finite functorial semi-norms in degree 1, 2 or 3 vanish on simply connected spaces: any integral homology class of degree 1, 2 or 3 of a simply connected space can be represented by a sphere.

Proposition 7.6. *For $d \in \mathbb{N}_{\geq 4}$ the following statements are equivalent:*

(1) *There is a* finite *functorial semi-norm $|\cdot|$ on $H_d(\,\cdot\,;\mathbb{R})$ such that for some homology class $\alpha \in H_d(X;\mathbb{R})$ of some simply connected space X we have $|\alpha| \neq 0$.*
(2) *There exists an oriented closed simply connected d-manifold that is strongly inflexible.*

Proof. First, let us assume that the first statement holds. Without loss of generality we may assume that X is path-connected and (in view of the triangle inequality) that α is rational. By Corollary 3.2 we can write $\alpha = a \cdot H_d(f;\mathbb{R})[M]_\mathbb{R}$, where M is some oriented closed simply connected d-manifold, $f\colon M \longrightarrow X$ is a continuous map, and $a \in \mathbb{R} \setminus \{0\}$. We now show that the manifold M is strongly inflexible: Because $|\cdot|$ is finite and functorial, we obtain

$$\infty > \big|[M]_\mathbb{R}\big| \geq \frac{1}{|a|} \cdot |\alpha| > 0.$$

If N is an oriented closed connected d-manifold, then for all continuous maps $g\colon N \longrightarrow M$ it follows that

$$|\deg(g)| \leq \frac{\big|[N]_\mathbb{R}\big|}{\big|[M]_\mathbb{R}\big|} < \infty.$$

Hence, $\deg(N, M)$ is finite, and so M is strongly inflexible.

Conversely, if there exists an oriented closed simply connected strongly inflexible d-manifold M, we consider the functorial semi-norm $|\cdot|$ associated with the domination semi-norm v_M for M (see Section 7.1). Because M is strongly inflexible, v_M is finite. So by Theorem 4.2, also $|\cdot|$ is finite, and $|[M]_\mathbb{R}| = v_M(M) = 1 \notin \{0, \infty\}$. □

Corollary 7.7 (Degrees 4, 5, and 6)**.** *All finite functorial semi-norms in degree* 4, 5, *and* 6 *vanish on simply connected spaces.*

Proof. All oriented closed connected simply connected manifolds of dimension at most 6 are flexible (Proposition 6.17), and so cannot be strongly inflexible. Hence, the claim follows by applying Proposition 7.6. □

Because finite functorial semi-norms vanish on flexible homology classes, we obtain:

Corollary 7.8. *Let X be a rational space whose minimal model is pure. Then every finite functorial semi-norm vanishes on every homology class of X in positive degree.*

Proof. This is a direct consequence of the fact that rational spaces with pure minimal model are almost flexible (Proposition 6.19). ☐

Moreover, it follows from Gaifullin's work [10] that finite functorial semi-norms which are multiplicative with respect to finite coverings are trivial on simply connected spaces:

Definition 7.9 (URC-manifold [9, p. 1747])**.** Let $d \in \mathbb{N}$. An oriented closed connected d-manifold M is a *URC-manifold* (Universal Realisation of Cycles), if for every topological space X and every $\alpha \in H_d(X;\mathbb{Z})$, there is a finite sheeted covering $\overline{M}$ of M, a map $f\colon \overline{M} \longrightarrow X$, and $k \in \mathbb{Z} \setminus \{0\}$ such that

$$H_d(f;\mathbb{Z})\bigl([\overline{M}]\bigr) = k \cdot \alpha \in H_d(X;\mathbb{Z}).$$

Gaifullin proved that there are many URC-manifolds in each dimension [9, Theorem 1.3]. Clearly, any URC-manifold of dimension at least 2 is strongly inflexible and has non-zero simplicial volume, because its finite coverings dominate hyperbolic manifolds, which are strongly inflexible (Example 6.15).

Example 7.10 (Functorial semi-norms associated with coverings of URC–manifolds)*.* Let $d \in \mathbb{N}$, let M be an oriented closed connected URC-manifold of dimension d, and let $S \subset \mathrm{Mfd}_d$ be the subclass of all finite connected covering spaces of M. Then $v_M|_S$ is a functorial semi-norm on S. If $d \geq 2$, then M is strongly inflexible, and so $v_M|_S$ is a finite functorial semi-norm on S with $v_M|_S(M) = 1$. More explicitly, multiplicativity under finite coverings and functoriality of simplicial volume show that

$$v_M|_S(N) = \frac{\|N\|}{\|M\|} = \text{number of sheets of any covering } N \to M$$

holds for all $N \in S$.

Let $|\cdot|_M^c$ be the associated functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$; because of the URC-property, this functorial semi-norm $|\cdot|_M^c$ is finite.

Proposition 7.11 (Multiplicative finite functorial semi-norms)**.** *Let $d \in \mathbb{N}$ and let $|\cdot|$ be a finite functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$ that is multiplicative with respect to finite coverings, i.e., satisfying: For all topological spaces X, all finite coverings $p\colon Y \longrightarrow X$ and all $\alpha \in H_d(Y;\mathbb{R})$ we have*

$$\bigl|H_d(p;\mathbb{R})(\alpha)\bigr| = \frac{1}{k} \cdot |\alpha|,$$

where k denotes the number of sheets of p. Then there exists a constant $c \in \mathbb{R}_{\geq 0}$ such that for all topological spaces X and all $\alpha \in H_d(X;\mathbb{R})$ we have

$$|\alpha| \leq c \cdot \|\alpha\|_1.$$

In particular, $|\cdot|$ is trivial on simply connected spaces.

Proof. Without loss of generality we may assume that $d \geq 2$. Let M be an oriented closed connected URC-manifold M in dimension d. It follows from the arguments of Gaifullin that there is a constant $a \in \mathbb{R}_{\geq 0}$ satisfying [10, Proposition 6.2]

$$|\cdot|^c_M \leq a \cdot \|\cdot\|_1.$$

On the other hand, multiplicativity of $|\cdot|$ and the construction of $|\cdot|^c_M$ (Example 7.10) show that

$$\big|[N]_\mathbb{R}\big| = \big|[M]_\mathbb{R}\big| \cdot \big|[N]_\mathbb{R}\big|^c_M$$

holds for all $N \in S$ and hence that $|\cdot| \leq |[M]_\mathbb{R}| \cdot |\cdot|^c_M$. Therefore,

$$|\cdot| \leq a \cdot \big|[M]_\mathbb{R}\big| \cdot \|\cdot\|_1.$$ □

8. Appendix I: Four inflexible Poincaré dgas

This appendix is devoted to the algebraic side of inflexibility – we construct the four inflexible Poincaré differential graded algebras used in Section 6. We explain the construction in Section 8.1. In Section 8.2, we prove that these dgas are Poincaré dgas; the intersection forms are computed in Section 8.3. In Section 8.4, we show that these dgas are inflexible.

8.1. **A design pattern for possibly inflexible dgas.** We start by defining a collection of dgas; all of the four concrete examples below follow the same design pattern based on two examples of Arkowitz and Lupton [2, Example 5.1 and 5.2], which are respectively examples A_3 and A_4 below. We shall construct dgas having the following properties:

– two generators x_1, x_2 of even degree with trivial differential,
– four generators y_1, y_2, y_3, z of odd degree; the differential is given by

$$\begin{aligned} dy_1 &:= x_1^3 x_2 \\ dy_2 &:= x_1^2 x_2^2 \\ dy_3 &:= x_1 x_2^3, \end{aligned}$$

and for the differential of z we choose $z' \in \bigwedge(x_1, x_2, y_1, y_2, y_4)$ in such a way that $d(y_1y_2y_3) = x_1^k \cdot z'$ or $d(y_1y_2y_3) = x_2^k \cdot z'$ and set

$$dz := z' + x_1^{k_1} + x_2^{k_2}$$

with suitable exponents k, k_1, $k_2 \in \mathbb{N}_{>0}$.

By construction, $d \circ d(y_j) = 0$ for all $j \in \{1, 2, 3\}$ and $d \circ d(z) = 0$; moreover, these dgas are finitely generated minimal dgas.

The following four examples dgas

$$A_j := \Big(\bigwedge(x_1, x_2, y_1, y_2, y_3, z), d\Big)$$

with $j \in \{1, \dots, 4\}$ are all of this kind:

Example 8.1 (A_1: an elliptic inflexible dga of formal dimension 64). Define the dga A_1 with generators of degrees

$$(|x_1|, |x_2|, |y_1|, |y_2|, |y_3|, |z|) = (2, 4, 9, 11, 13, 35)$$

where the differential d is given by

$$\begin{array}{lllllllll} dx_1 & := & 0 & dy_1 & := & x_1^3x_2 & dz & := & x_2^4y_1y_2 - x_1x_2^3y_1y_3 + x_1^2x_2^2y_2y_3 \\ dx_2 & := & 0 & dy_2 & := & x_1^2x_2^2 & & + & x_1^{18} + x_2^9 \\ & & & dy_3 & := & x_1x_2^3 & & = & x_2^2 \cdot w + x_1^{18} + x_2^9, \end{array}$$

where we use the abbreviation $w := x_2^2y_1y_2 - x_1x_2y_1y_3 + x_1^2y_2y_3$; in other words $x_1x_2w = d(y_1y_2y_3)$.

Example 8.2 (A_2: an elliptic inflexible dga of formal dimension 108). Define the dga A_2 with generators of degrees

$$(|x_1|,|x_2|,|y_1|,|y_2|,|y_3|,|z|) = (4,6,17,19,21,59)$$

where the differential d is given by

$$\begin{array}{lllllllll} dx_1 & := & 0 & dy_1 & := & x_1^3x_2 & dz & := & x_2^4y_1y_2 - x_1x_2^3y_1y_3 + x_1^2x_2^2y_2y_3 \\ dx_2 & := & 0 & dy_2 & := & x_1^2x_2^2 & & + & x_1^{15} + x_2^{10}. \\ & & & dy_3 & := & x_1x_2^3 & & & \end{array}$$

Example 8.3 (A_3: an elliptic inflexible dga of formal dimension 208 [2, Example 5.1]). Define the dga A_3 with generators of degrees

$$(|x_1|,|x_2|,|y_1|,|y_2|,|y_3|,|z|) = (8,10,33,35,37,119)$$

where the differential d is given by

$$\begin{array}{lllllllll} dx_1 & := & 0 & dy_1 & := & x_1^3x_2 & dz & := & x_1^4x_2^2y_1y_2 - x_1^5y_1y_3 + x_1^6y_2y_3 \\ dx_2 & := & 0 & dy_2 & := & x_1^2x_2^2 & & + & x_1^{15} + x_2^{12}. \\ & & & dy_3 & := & x_1x_2^3 & & & \end{array}$$

Example 8.4 (A_4: an elliptic inflexible dga of formal dimension 228 [2, Example 5.2]). Define the dga A_4 with generators of degrees

$$(|x_1|,|x_2|,|y_1|,|y_2|,|y_3|,|z|) = (10,12,41,43,45,119)$$

where the differential d is given by

$$\begin{array}{lllllllll} dx_1 & := & 0 & dy_1 & := & x_1^3x_2 & dz & := & x_2^3y_1y_2 - x_1x_2^2y_1y_3 + x_1^2x_2y_2y_3 \\ dx_2 & := & 0 & dy_2 & := & x_1^2x_2^2 & & + & x_1^{12} + x_2^{10}. \\ & & & dy_3 & := & x_1x_2^3 & & & \end{array}$$

We will carry out the proofs in detail only for the dga A_1 defined in Example 8.1 – in fact, this is the most complicated of the four examples and the other examples can be treated by analogous arguments and calculations.

8.2. **The example dgas are Poincaré dgas.** Recall that a minimal Sullivan algebra $(\bigwedge V, d)$ is called *elliptic* if V and $H^*(\bigwedge V, d)$ are finite dimensional.

Proposition 8.5 (Ellipticity)**.** *The dgas A_1, A_2, A_3, and A_4 are elliptic.*

Proof. As these dgas are finitely generated by construction, it suffices to show that their cohomology is finite dimensional. In other words, it suffices to show that the cohomology is generated by nilpotent classes. Because the odd degree generators are nilpotent on the level of the dgas and because the differential is trivial on the even degree generators, it suffices to show that the classes $[x_1]$ and $[x_2]$ are nilpotent.

We now show that $[x_1]$ and $[x_2]$ are nilpotent in $H^*(A_1)$ (the arguments for the other example dgas are similar): By definition of d, we have

$$[x_1]^{19} = [x_1^{19}] = \big[x_1 dz - x_2 d(y_1y_2y_3) - x_1 x_2^9\big] = \big[d(x_1 z - x_2 y_1 y_2 y_3 - x_2 y_3)\big] \\ = 0.$$

Therefore we obtain

$$\begin{aligned}[x_2]^{18} &= [x_2^9]\cdot[x_2^9] = \big([dz - x_2^2 w - x_1^{18}]\big)^2 = \big([x_2^2 w] - [x_1^{18}]\big)^2 \\ &= \big[(x_2^2 w)^2\big] - 2[x_1^{18} x_2^2 w] + [x_1]^{36} = 0 - 2[d(x_1^{17} x_2 y_1 y_2 y_3)] + 0 \\ &= 0;\end{aligned}$$

notice that $w^2 = 0$ because every summand of w contains two of the three odd generators y_1, y_2, y_3 and $y_j^2 = 0$. □

We will now select non-zero classes in the top cohomology, which will play the rôle of fundamental classes:

Proposition 8.6 (Fundamental classes for A_1, A_2, A_3, A_4)**.**

(1) *The class* $[x_2]^{16}$ *is non-zero in* $H^{64}(A_1)$.
(2) *The class* $[x_2]^{18}$ *is non-zero in* $H^{108}(A_2)$.
(3) *The class* $[x_1]^{26}$ *is non-zero in* $H^{208}(A_3)$.
(4) *The class* $[x_2]^{19}$ *is non-zero in* $H^{228}(A_4)$.

Proof. We give the proof only for A_1, the other cases being similar. *Assume* for a contradiction that $[x_2^{16}] = 0$ in $H^{64}(A_1)$; hence, there is an element u of A_1 of degree 63 with $du = x_2^{16}$. We can write u as

$$u = pz + p_{12} y_1 y_2 z + p_{13} y_1 y_3 z + p_{23} y_2 y_3 z + p_1 y_1 + p_2 y_2 + p_3 y_3,$$

where $p, p_{12}, p_{13}, p_{23}, p_1, p_2, p_3$ are homogeneous polynomials in x_1, x_2.
Then

$$\begin{aligned} x_2^{16} = du = {} & p x_2^9 + p x_1^{18} + p x_2^4 y_1 y_2 - p x_1 x_2^3 y_1 y_3 + p x_1^2 x_2^2 y_2 y_3 \\ & + p_{12} d(y_1 y_2) z + p_{12} x_2^9 y_1 y_2 + p_{12} x_1^{18} y_1 y_2 + 0 \\ & + p_{13} d(y_1 y_3) z + p_{13} x_2^9 y_1 y_3 + p_{13} x_1^{18} y_1 y_3 + 0 \\ & + p_{23} d(y_2 y_3) z + p_{23} x_2^9 y_2 y_3 + p_{23} x_1^{18} y_2 y_3 + 0 \\ & + q, \end{aligned}$$

where q is a homogeneous polynomial in x_1, x_2 that is divisible by $x_1 x_2$; the zeroes at the end of the lines stem from the fact that squares of odd degree elements are zero and each summand of w contains two of the three odd degree generators y_1, y_2, y_3.

Because A_1 is freely generated by $x_1, \dots, z$, comparing the x_2^{16}-coefficients on both sides shows that $p \neq 0$. Moreover, comparing the z-coefficients of both sides yields

$$p_{12} d(y_1 y_2) + p_{13} d(y_1 y_3) + p_{23} d(y_2 y_3) = 0.$$

Comparing the coefficients of y_1, y_2, y_3 in this equation gives us

$$\begin{aligned} -x_1^2x_2^2p_{12} &= x_1x_2^3p_{13} \\ x_1^3x_2p_{12} &= x_1x_2^3p_{23} \\ x_1^3x_2p_{13} &= -x_1^2x_2^2p_{23}. \end{aligned}$$

Because $\deg p_{12} = 8$, $\deg p_{13} = 6$, and $\deg p_{23} = 4$, a simple divisibility argument shows that there is an $\eta \in \mathbb{Q}$ with

$$p_{13} = -\eta \cdot x_1x_2, \qquad p_{23} = \eta \cdot x_1^2, \qquad p_{12} = \eta \cdot x_2^2.$$

Hence, comparing the summands of du that are divisible by y_1y_2, but not by z, shows that

$$0 = px_2^4 + p_{12}x_2^9 + p_{12}x_1^{18} = px_2^4 + \eta \cdot x_2^{11} + \eta \cdot x_1^{18}x_2^2.$$

Because $p \neq 0$, it follows that $\eta = 0$ (otherwise the last summand is not divisible by x_2^4). On the other hand, by an analogous argument, we obtain

$$0 = -px_1x_2^3 + p_{13}x_2^9 + p_{13}x_1^{18} = -px_1x_2^3 - \eta \cdot x_1x_2^{10} - \eta \cdot x_1^{19}x_2,$$

and thus $p = 0$, contradicting $p \neq 0$. So x_2^{16} cannot be a coboundary. □

Corollary 8.7 (The dgas $A_1, \dots, A_4$ are Poincaré dgas)**.** *The dgas $A_1, \dots, A_4$ are Poincaré dgas with the cohomology classes in Proposition 8.6 as fundamental classes.*

Proof. The dgas $A_1, \dots, A_4$ are elliptic (Proposition 8.5). By a classical result in rational homotopy theory [7, Proposition 38.3], cohomology algebras of elliptic minimal Sullivan algebras are Poincaré duality algebras; clearly, any non-zero cohomology class in the top cohomology can be chosen as fundamental class. □

8.3. **The intersection forms of the example dgas.**

Proposition 8.8 (Intersection form of A_1)**.** *The classes $[x_2w]$, $[x_1^2w]$, $[x_1^{16}]$, and $[x_2^8]$ form a $\mathbb{Q}$-basis of $H^{32}(A_1)$ (the middle cohomology of A_1), and the intersection form with respect to this basis and the fundamental class $[x_2^{16}]$ of A_1 (see Proposition 8.6) looks as follows:*

$$\begin{pmatrix} 0^3 & 0^3 & 0^6 & -1^7 \\ 0^3 & 0^3 & 1^4 & 0^5 \\ 0^6 & 1^4 & 0^2 & 0^1 \\ -1^7 & 0^5 & 0^1 & 1^0 \end{pmatrix}$$

(The superscripts in the matrix refer to the part of the proof where the corresponding coefficient is computed).

Proof. We first show that $H^{32}(A_1)$ is generated by $[x_2w]$, $[x_1^2w]$, $[x_1^{16}]$, and $[x_2^8]$: What do cocycles of degree 32 in A_1 look like? Clearly, x_1^{16} and x_2^8 are cocycles of degree 32. All cocycles in the subalgebra $\bigwedge(x_1, x_2)$ divisible by x_1x_2 are in the image of d (by definition of dy_1, dy_2, dy_3). Because the differential is trivial on $\bigwedge(x_1, x_2)$, it remains to look at cocycles of the form

$$u = p_{12}y_1y_2 + p_{13}y_1y_3 + p_{23}y_2y_3,$$

where $p_{12}, p_{13}, p_{23} \in \bigwedge(x_1, x_2)$. If $du = 0$, then looking the coefficients of y_1, y_2, and y_3 respectively in du leads to

$$x_1^3 x_2 p_{12} = x_1 x_2^3 p_{23}$$
$$x_1^2 x_2^2 p_{12} = -x_1 x_2^3 p_{13}$$
$$x_1^3 x_2 p_{13} = -x_1^2 x_2^2 p_{23};$$

As $\deg p_{12} = 12$, $\deg p_{13} = 10$, and $\deg p_{23} = 8$, a simple divisibility consideration shows that there exist $\eta_1, \eta_2 \in \mathbb{Q}$ such that

$$p_{23} = \eta_1 \cdot x_1^4 + \eta_2 \cdot x_1^2 x_2$$
$$p_{12} = \eta_1 \cdot x_1^2 x_2^2 + \eta_2 \cdot x_2^3$$
$$p_{13} = -\eta_1 \cdot x_1^3 x_2 - \eta_2 \cdot x_1 x_2^2;$$

hence, $u = \eta_1 \cdot x_1^2 w + \eta_2 \cdot x_2 w$. So $H^{32}(A_1)$ indeed is generated as $\mathbb{Q}$-vector space by $[x_2 w]$, $[x_1^2 w]$,$[x_1^{16}]$, and $[x_2^8]$.

As next step, we determine the corresponding matrix for the intersection form with respect to the fundamental class $[x_2^{16}]$:

(0) Because we chose $[x_2^{16}]$ as fundamental class with respect to which the intersection form is computed, the matrix coefficient corresponding to column $[x_2^8]$ and row $[x_2^8]$ equals 1.
(1) We have $[x_1^{16}] \cdot [x_2^8] = [d(x_1^{13} x_2^7 y_1)] = 0$.
(2) Moreover, $[x_1^{16}] \cdot [x_1^{16}] = [x_1^{32}] = 0$ as was shown in the proof of Proposition 8.5.
(3) Squares of elements of A_1 of odd degree are zero, and hence $w^2 = 0$ (because each summand of w contains two of the three odd elements y_1, y_2, y_3).
(4) We have (because $(dz)w = 0 = ww$ as in the previous item)

$$\begin{aligned}[x_1^{16}] \cdot [x_1^2 w] &= [(dz) \cdot w - x_2^9 w - x_2^2 ww] \\ &= [0 - x_2^9 w - 0] \\ &= -[d(x_2^7 z) + x_2^{16} + x_2^7 x_1^{18}] \\ &= -[x_2^{16}] + [d(x_1^{15} x_2^6 y_1)] \\ &= -[x_2^{16}].\end{aligned}$$

(5) Moreover, $[x_2^8] \cdot [x_1^2 w] = [d(x_2^7 x_1 y_1 y_2 y_3)] = 0$.
(6) Analogously, $[x_1^{16}] \cdot [x_2 w] = [d(x_1^{15} y_1 y_2 y_3)] = 0$.
(7) Finally,

$$\begin{aligned}[x_2^8] \cdot [x_2 w] &= [d(x_2^7 z) - x_1^{18} x_2^7 - x_2^{16}] \\ &= [-d(x_1^{15} x_2^6 y_1) - x_2^{16}] \\ &= -[x_2^{16}].\end{aligned}$$

Moreover, from the shape of this matrix we can easily deduce that the elements $[x_2 w]$, $[x_1^2 w]$,$[x_1^{16}]$, and $[x_2^8]$ are linearly independent over $\mathbb{Q}$. □

Remark 8.9 (Intersection form of A_2, A_3, A_4). Similarly to the previous proposition one can show that:

– The classes $[x_2^3 y_1 y_2 - x_1 x_2^2 y_1 y_3 + x_1^2 x_2 y_2 y_3], [x_2]^8$ form a $\mathbb{Q}$-basis of the middle cohomology $H^{54}(A_2)$ of A_2. The intersection form of A_2 with respect to this basis and the fundamental class $[x_2]^{18}$ of A_2 is

$$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

– The classes $[x_1^2 x_2^2 y_1 y_2 - x_1^3 x_2 y_1 y_3 + x_1^4 y_2 y_3], [x_1]^{13}$ form a $\mathbb{Q}$-basis of the middle cohomology $H^{104}(A_3)$ of A_3. The intersection form of A_3 with respect to this basis and the fundamental class $[x_1]^{26}$ is

$$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

– The middle cohomology $H^{114}(A_4)$ of A_4 is zero.

8.4. **The example dgas are inflexible.** We now show that the four Poincaré dgas A_1, A_2, A_3, and A_4 are *inflexible* in the sense that there is no dga morphism whose induced homomorphism on cohomology maps the fundamental class to a non-trivial multiple of itself.

Proposition 8.10 (Inflexibility). *The dgas A_1, A_2, A_3, and A_4 are inflexible.*

Proof. We will give the complete calculation only for the example A_1; for the other dgas the calculation is similar, and even a bit simpler as the degrees of the even generators x_1 and x_2 are less entangled; moreover, for the dgas A_3 and A_4 an argument is contained in the work of Arkowitz and Lupton [2, Examples 5.1 and 5.2].

Let $f\colon A_1 \longrightarrow A_1$ be a dga morphism; looking at the degrees of the generators of A_1 we see that there are constants $\alpha_1, \alpha_2, \alpha_{2,1}, \dots, \gamma, \gamma_1 \in \mathbb{Q}$ and homogenous polynomials p_1, p_2, p_3 in x_1, x_2 such that

$$\begin{aligned} f(x_1) &= \alpha_1 \cdot x_1 \\ f(x_2) &= \alpha_2 \cdot x_2 + \alpha_{2,1} \cdot x_1^2 \\ f(y_1) &= \beta_1 \cdot y_1 \\ f(y_2) &= \beta_2 \cdot y_2 + \beta_{2,1} \cdot x_1 y_1 \\ f(y_3) &= \beta_3 \cdot y_3 + \beta_{3,1} \cdot x_1^2 y_1 + \beta_{3,2} \cdot x_2 y_1 + \beta_{3,3} \cdot x_1 y_2 \\ f(z) &= \gamma \cdot z + \gamma_1 \cdot x_1 y_1 y_2 y_3 + p_1 y_2 + p_2 y_2 + p_3 y_3. \end{aligned}$$

Using that f as a dga morphism is compatible with the differential d of A_1 and that A_1 is freely generated by $x_1, \dots, z$, we deduce constraints on the coefficients $\alpha_1, \dots$. Notice that because we chose $[x_2^{16}]$ as fundamental class of A_1, we can read off the "degree" of f from the coefficient α_2, and it suffices to show that $\alpha_2 \in \{-1, 0, 1\}$.

(1) Comparing the coefficients for $f \circ d(y_1)$ and $d \circ f(y_1)$, we obtain

$$\beta_1 = \alpha_1^3 \alpha_2$$

and $\alpha_1^3 \alpha_{2,1} = 0$. In particular, $\alpha_1 = 0$ or $\alpha_{2,1} = 0$.

(2) Comparing the coefficients for $f \circ d(y_2)$ and $d \circ f(y_2)$, we obtain in addition that

$$\beta_2 = \alpha_1^2 \alpha_2^2.$$

(3) Moreover, we have

$$f \circ d(z) = \alpha_1^{18} \cdot x_1^{18} + (\alpha_2 \cdot x_2 + \alpha_{2,1} \cdot x_1^2)^9 + \alpha_2^4 \beta_1 \beta_2 \cdot x_2^4 y_1 y_2 + q,$$
$$d \circ f(z) = \gamma \cdot x_1^{18} + \gamma \cdot x_2^9 + \gamma \cdot d(x_2 w) + \gamma_1 \cdot d(x_1 y_1 y_2 y_3),$$

where $q \in (x_1 x_2) \cdot A_1$. Comparing the coefficients of these elements shows that

$$\alpha_1^{18} + \alpha_{2,1}^9 = \gamma = \alpha_2^9.$$

Because $\gamma \cdot d(x_2 w) + \gamma_1 \cdot d(x_1 y_1 y_2 y_3)$ and q are divisible by $x_1 x_2$, it follows that

$$\gamma = \alpha_2^4 \beta_1 \beta_2 = \alpha_2^7 \alpha_1^5$$

(in the second equation we used the results from Steps 1 and 2).

In view of Step 1 we can assume that $\alpha_1 = 0$ or $\alpha_{2,1} = 0$. If $\alpha_1 = 0$, then also $\alpha_2^9 = \gamma = \alpha_2^7 \alpha_1^5 = 0$ by Step 3. On the other hand, if $\alpha_1 \neq 0$ and $\alpha_{2,1} = 0$, then

$$\alpha_1^{18} = \gamma = \alpha_2^9 \qquad \text{and} \qquad \alpha_2^7 \alpha_1^5 = \gamma = \alpha_2^9$$

by Step 3. Now a small computation shows that $\alpha_2 = 1$. Hence, A_1 is inflexible. □

9. Appendix II: more inflexible dgas and manifolds

In this appendix we produce more examples of inflexible manifolds from the basic examples of Section 6 and 8: Using connected sums and products, we obtain in infinitely many dimensions infinitely many homotopy types of oriented closed simply connected inflexible manifolds (Section 9.1 and Section 9.2). Moreover, we show that inflexibility is "generic" in the sense that in infinitely many dimensions, every oriented bordism class can be rationally represented by a simply connected inflexible manifold (Section 9.3).

Recall that if $(A, [A])$ is a Poincaré dga (Definition 6.4) then $\mathcal{M}(A, [A])$ denotes the class of all oriented closed simply connected manifolds that have trivial total Pontryagin class and realise this rational data (Definiton 6.7).

9.1. **Inflexible connected sums.**

In general, it is not clear that connected sums of inflexible manifolds are inflexible; however, in certain cases inflexibility is preserved under connected sums:

Theorem 9.1 (Inflexible connected sums)**.** *Let M be an oriented closed simply connected n-manifold with inflexible minimal model and $\pi_{n-1}(M) \otimes \mathbb{Q} = 0$. Suppose that $N_1, \dots, N_r$ are oriented closed simply connected n-manifolds such that $\deg(N_{j,\mathbb{Q}}, M_{\mathbb{Q}})$ is finite for each $j \in \{1, \dots, r\}$. Then the iterated connected sum*

$$M \# N_1 \# \cdots \# N_r$$

is inflexible. In particular, for all $r \in \mathbb{N}$ the r-fold connected sum $M^{\#r}$ is inflexible.

The proof of this theorem relies on applying repeatedly the following lemma: Recall that $\deg(N, M)$ is the set of degrees of maps between oriented closed connected manifolds N and M; also, for subsets $A, B \subset \mathbb{Z}$ we write

$$A + B := \{a + b \mid a \in A,\ b \in B\} \subset \mathbb{Z}.$$

Lemma 9.2. *Let N_1, N_2 and M be oriented closed simply connected n-manifolds with rationalisations $N_{1,\mathbb{Q}}$, $N_{2,\mathbb{Q}}$, and $M_\mathbb{Q}$. If $\pi_{n-1}(M_\mathbb{Q}) = 0$ then*

$$\deg(N_1 \# N_2, M) \subset \deg(N_{1,\mathbb{Q}}, M_\mathbb{Q}) + \deg(N_{2,\mathbb{Q}}, M_\mathbb{Q})\,.$$

Proof. Because rationalisation preserves rational cohomology, we have

$$\deg(N_1 \# N_2, M) \subset \deg\bigl((N_1 \# N_2)_\mathbb{Q}, M_\mathbb{Q}\bigr);$$

so it suffices to show $\deg\bigl((N_1 \# N_2)_\mathbb{Q}, M_\mathbb{Q}\bigr) \subset \deg(N_{1,\mathbb{Q}}, M_\mathbb{Q}) + \deg(N_{2,\mathbb{Q}}, M_\mathbb{Q})$. To this end, we consider the cofibration sequence

$$(*) \qquad S^{n-1} \longrightarrow N_1 \# N_2 \longrightarrow (N_1 \# N_2) \cup_{S^{n-1}} D^n,$$

where we attach D^n along the inclusion $i\colon S^{n-1} \longrightarrow N_1 \# N_2$ where S^{n-1} is the locus of the connected sum between N_1 and N_2. Clearly, the space $W := \bigl((N_1 \# N_2) \cup_{S^{n-1}} D^n\bigr)$ is homotopic to the wedge $N_1 \vee N_2$: we will use this fact below. From the cofibration sequence $(*)$ and its rationalisation we obtain the following commutative diagram of exact sequences:

$$(**) \qquad \begin{array}{ccccc} [W, M] & \longrightarrow & [N_1 \# N_2, M] & \longrightarrow & [S^{n-1}, M] \\ \big\downarrow{\cdot_\mathbb{Q}} & & \big\downarrow{\cdot_\mathbb{Q}} & & \big\downarrow{\cdot_\mathbb{Q}} \\ [W_\mathbb{Q}, M_\mathbb{Q}] & \longrightarrow & [(N_1 \# N_2)_\mathbb{Q}, M_\mathbb{Q}] & \longrightarrow & [S^{n-1}_\mathbb{Q}, M_\mathbb{Q}] \end{array}$$

The lower sequence can be seen to be exact by looking at a concrete description of Sullivan models of cell additions (up to quasi-isomorphism) [7, Diagram 13.15].

But $[S^{n-1}_\mathbb{Q}, M_\mathbb{Q}] \cong \pi_{n-1}(M_\mathbb{Q}) = 0$ by assumption. Thus, up to homotopy every map from the connected sum $(N_1 \# N_2)_\mathbb{Q} \longrightarrow M_\mathbb{Q}$ factors through the map $(N_1 \# N_2)_\mathbb{Q} \longrightarrow W_\mathbb{Q}$ induced by the inclusion. We observed above that there is a homotopy equivalence $W \simeq N_1 \vee N_2$. The characterisation of rationalisations in terms of singular homology with integral coefficients [7, Theorems 9.3 and 9.6] together with the Mayer-Vietoris sequence in homology show that $W_\mathbb{Q} \simeq N_{1,\mathbb{Q}} \vee N_{2,\mathbb{Q}}$. Hence we have the equality

$$[W_\mathbb{Q}, M_\mathbb{Q}] = [N_{1,\mathbb{Q}}, M_\mathbb{Q}] \times [N_{2,\mathbb{Q}}, M_\mathbb{Q}].$$

So from the commutative diagram $(**)$ above we see that there is an inclusion $\deg((N_1 \# N_2)_\mathbb{Q}, M_\mathbb{Q})) \subset \deg(N_{1,\mathbb{Q}}, M_\mathbb{Q}) + \deg(N_{2,\mathbb{Q}}, M_\mathbb{Q})$, as desired. ☐

Proof of of Theorem 9.1. For $N := M \# N_1 \# \cdots \# N_r$, observe that the obvious collapse map $N \longrightarrow M$ has degree 1 and so $1 \in \deg(N, M)$. Applying Lemma 9.2 inductively we conclude that $\deg(N, M)$ is finite, since we have assumed that the sets $\deg(N_{j,\mathbb{Q}}, M_\mathbb{Q})$ and $\deg(M_\mathbb{Q}, M_\mathbb{Q})$ are finite. But the monoid $\mathrm{Map}(N, N)$ of self maps of N acts by pre-composition on the set $\mathrm{Map}(N, M)$ of maps from N to M and since $1 \in \deg(N, M)$ we see that there is an inclusion $\deg(N, N) \subset \deg(N, M)$. Hence $\deg(N, N)$ is finite and N is inflexible. ☐

In order to apply Theorem 9.1 to our examples we shall need information about the group $\pi_{\dim M - 1}(M) \otimes \mathbb{Q}$ for $M \in \mathcal{M}(A_j, [A_j])$ with $j = \{1, \dots, 4\}$, where $A_1, \dots, A_4$ are the dgas from Section 8. Recall that $\pi_*(M) \otimes \mathbb{Q}$ is a

$\mathbb{Q}$-vector space generated by the indecomposable elements of the minimal model of M [7, Theorem 15.11]. Using the notation of Section 8.1, it follows that

$$\pi_*(M) \otimes \mathbb{Q} \cong \mathbb{Q}(x_1) \oplus \mathbb{Q}(x_2) \oplus \mathbb{Q}(y_1) \oplus \mathbb{Q}(y_2) \oplus \mathbb{Q}(y_3) \oplus \mathbb{Q}(z).$$

The degrees of the generators x_1, x_2, y_1, y_2, y_3, z for each of the A_j's are listed in Section 8.1. In particular we obtain:

Lemma 9.3. *For all $j \in \{1, \ldots, 4\}$ and for all $M \in \mathcal{M}(A_j, [A_j])$, we have that $\pi_{n-1}(M) \otimes \mathbb{Q} = 0$ where n is the dimension of M.*

Theorem 9.1 allows us to prove the existence of large classes of inflexible manifolds. We do this systematically in Section 9.3 and for now present the following simple example:

Example 9.4. Let $j \in \{1, \ldots, 4\}$, let $M \in \mathcal{M}(A_j, [A_j])$, and let $r \in \mathbb{N}_{>0}$. Then the r-fold connected sum $M^{\#r}$ is inflexible (and simply connected). Looking at the rational cohomology ring of these manifolds shows that $M^{\#r} \not\simeq M^{\#s}$, whenever $r \neq s$.

9.2. **Inflexible products.** In general, it is not clear that products of inflexible manifolds are inflexible as maps between products of manifolds cannot necessarily be decomposed into maps on the factors; we will show now that certain products of our basic examples of simply connected manifolds are inflexible:

Theorem 9.5 (Inflexible products)**.** *Let $j \in \{2, 3, 4\}$, let $M \in \mathcal{M}(A_j, [A_j])$ be a manifold as in Theorem 6.8, and let $k \in \mathbb{N}_{>0}$. Then the k-fold product $M^{\times k}$ is inflexible (and simply connected).*

This result is proved in the following by carefully analysing the algebraic counterpart, namely tensor products of the Poincaré dgas A_2, A_3, and A_4 respectively: Recall that given dgas A and B there is the tensor product dga $A \otimes B$ [7, Example 3 on p. 47] and that $H^*(A \otimes B) \cong H^*(A) \otimes H^*(B)$. In particular, if $(A, [A])$ and $(B, [B])$ are Poincaré dgas then so is the product $(A \otimes B, [A] \otimes [B])$.

Proposition 9.6. *For each $j \in \{2, 3, 4\}$ and for all $k \in \mathbb{N}_{>0}$ the k-fold tensor product $(A_j^{\otimes k}, [A_j]^{\otimes k})$ is an inflexible Poincaré dga.*

Proof. We shall give the proof for A_3 and then state the modifications necessary for A_2 and A_4. Let us fix some notation: for an index $a \in \{1, \ldots, k\}$ let A_{3a} denote the a-th copy of A_3 in the k-fold product $A_3^{\otimes k}$. Similarly, for generators $x_i, y_i \in A_3$ as in Section 8.1 let x_{ia} and y_{ia} denote the copy of x_i or y_i in A_{3a}. Notice that because the fundamental class of each A_{3a} is given by $[x_{1a}]^{26}$ the fundamental class of $A_3^{\otimes k}$ is given by $\otimes_{a=1}^{k}[x_{1a}]^{26}$. Therefore, we can read off the degree of dga endomorphisms of $A_3^{\otimes k}$ by looking at the situation in degree $|x_1| = 8$.

Now let $f\colon A_3^{\otimes k} \longrightarrow A_3^{\otimes k}$ be a dga endomorphism of non-zero degree. Since $A_3^{\otimes k}$ is Poincaré with finite dimensional cohomology it follows that f induces isomorphisms on all cohomology groups, and so f is a dga isomorphism [7, Proposition 12.10(i)]. Thus, $H^8(f)\colon H^8(A_3^{\otimes k}) \longrightarrow H^8(A_3^{\otimes k})$

is a $\mathbb{Q}$-linear isomorphism. By construction of A_3, there is a canonical isomorphism $(A_3^{\otimes k})^8 \cong H^8(A_3^{\otimes k})$, which identifies $H^8(f)$ with $f|_{(A_3^{\otimes k})^8}$. In particular, also $f|_{(A_3^{\otimes k})^k}$ is a $\mathbb{Q}$-linear isomorphism.

We shall show below that $H^8(f)$ is represented by a signed permutation matrix with respect to the obvious basis of $(A_3^{\otimes k})^8 = (A_3^8)^{\oplus k}$. If this holds, then, because $\otimes_{a=1}^k [x_{1a}]^{26}$ is a fundamental class of $A_3^{\otimes k}$, the dga map f has degree 1 or -1, which proves that $A_3^{\times k}$ is inflexible.

In order to complete the proof it therefore remains to prove that $H^8(f)$ is represented by a signed permutation matrix: For each $b \in \{1, \dots, k\}$ we have the dga projection $p_b \colon A_3^{\otimes k} \longrightarrow A_{3b}$ and the dga inclusion $i_b \colon A_{3b} \longrightarrow A_3^{\otimes k}$. Moreover, for $a, b \in \{1, \dots, k\}$ we consider the dga map

$$f_{ab} := p_a \circ f \circ i_b \colon A_{3b} \longrightarrow A_{3a}.$$

Since $A_{3a} = A_3 = A_{3b}$, we have by Proposition 8.10 that f_{ab} has degree 0, 1 or -1. Because $[A_3] = [x_1]^{26}$ and $A_3^8 = \mathbb{Q} \cdot x_1$ it follows that $f_{ab}(x_{1a}) = \pm x_{1b}$ or $f_{ab}(x_{1a}) = 0$. Thus, for all $a \in \{1, \dots, k\}$ we obtain

$$(*) \qquad f(x_{1a}) = \sum_{b=1}^{k} \varepsilon_{ab} \cdot x_{1b}, \quad \text{where } \varepsilon_{ab} \in \{-1, 0, 1\}.$$

We proceed now by contradiction: Suppose that for some a at least two of the coefficients $\{\varepsilon_{ab} \mid b \in \{1, \dots, k\}\}$ are non-zero. By construction of A_3 for $i \in \{10, 33\}$ there are identifications $(A_3^{\otimes k})^i = \bigoplus_{a=1}^k A_{3a}^i$. We now consider the equation

$$df(y_{1a}) = f(dy_{1a}).$$

The left hand side is a sum of monomials of the form $x_{1c}^3 x_{2c}$, which can be seen by looking at the definition of A_{3a}^{33} and of the differential on A_3 (Example 8.3). However, on the right hand side, we have $f(dy_{1a}) = f(x_{1a}^3 x_{2a}) = f(x_{1a})^3 \cdot f(x_{2a})$. Using the description of $f(x_{1a})$ from Equation $(*)$ and the fact that there are two non-zero coefficients ε_{ab} and $\varepsilon_{ab'}$, it follows that the right hand side contains monomials of the form $\pm C_{bb'c} \cdot x_{1b}^2 \cdot x_{1b'} \cdot x_{2c}$ where $b \neq b'$ and $C_{bb'c} \in \mathbb{Q} \setminus \{0\}$. But such monomials are not present on the left hand side, which is a contradiction. Therefore, we can conclude that for each $a \in \{1, \dots, k\}$ only one of the coefficients $\varepsilon_{ab} \in \{-1, 0, 1\}$ is non-zero. As $H^8(f) = f|_{(A_3^{\otimes k})^8}$ is an isomorphism, it follows that $H^8(f)$ indeed is represented by a signed permutation matrix.

For the dgas A_2 and A_4 the fundamental class is a power of x_2 and so we repeat the line of argument this time using $(A_2^{\otimes k})^{10} = \oplus_{a=1}^k A_{2a}^{10}$ or $(A_4^{\otimes k})^{12} = \oplus_{a=1}^k A_{4a}^{12}$ and the equation $dy_{2a} = x_{1a}^2 x_{2a}^2$ instead. □

Notice that the above proof does not directly carry over to the case of the Poincaré dga $(A_1, [x_2]^{16})$ because dga endomorphisms of A_1 are slightly more complicated in degree $|x_2| = 4$ than in the cases discussed above.

Proof of Theorem 9.5. The minimal model of $M^{\times k}$ is the k-fold tensor product $A_j^{\otimes k}$ [7, Example 1 p. 248]; moreover, the fundamental class of M^k corresponds to $[A_j]^{\otimes k} \in A_j^{\otimes k}$. Now the theorem follows because the Poincaré dga $A_j^{\otimes k}$ is inflexible by Proposition 9.6. □

Corollary 9.7. *In each of infinitely many dimensions there exist infinitely many rational homotopy types of oriented closed simply connected inflexible manifolds.*

Proof. Let $j \in \{2,3,4\}$, let $k \in \mathbb{N}_{>0}$, and let $r \in \mathbb{N}_{>0}$. Moreover, let $M \in \mathcal{M}(A_j, [A_j])$. Theorem 9.5 and Theorem 9.1 (together with Lemma 9.3) show that the oriented closed simply connected manifold $(M^{\times k})^{\#r}$ is inflexible. The rational cohomology of these manifolds shows that if $r \neq r'$, then $(M^{\times k})^{\#r}$ and $(M^{\times k})^{\#r'}$ do not have the same rational homotopy type. ☐

9.3. **Evidence for the genericity of inflexibility.** In the following, we combine results of the preceding sections to exhibit large numbers of simply connected inflexible manifolds: On the one hand, we show that there are "many" homotopy types of simply connected inflexible manifolds, and in particular that in many dimensions simply connected manifolds are "generic" from the point of view of oriented rational bordism. On the other hand, we show that simply connected inflexible manifolds exist that satisfy tangential structure constraints such as being parallelisable or non-spinable.

One way to create many (integral) homotopy types of simply connected inflexible manifolds out of a single inflexible Poincaré dga is to rescale the fundamental class of the dga in question:

Proposition 9.8 (Scaling the fundamental class)**.** *Let $(A, [A])$ be an inflexible Poincaré dga, and let $a, a' \in \mathbb{Q} \setminus \{0\}$ with $|a| \neq |a'|$. If $M \in \mathcal{M}(A, a \cdot [A])$ and $M' \in \mathcal{M}(A, a' \cdot [A])$, then $M \not\simeq M'$.*

Proof. Recall that any Poincaré dga is the minimal model of some simply connected rational $\mathbb{Q}$-Poincaré space (cf. proof of Proposition 6.9); hence there is a rational $\mathbb{Q}$-Poincaré space $(X, [X])$ realising $(A, [A])$.

Let $M \in \mathcal{M}(A, a \cdot [A])$ and $M' \in \mathcal{M}(A, a' \cdot [A])$; then the rationalisation of both M and M' coincides with X, the only difference being that the fundamental classes are mapped to different multiples of $[X]$. Let $\rho_M \colon M \longrightarrow M_\mathbb{Q} = X$ and $\rho_{M'} \colon M' \longrightarrow X$ be the canonical maps provided by the rationalisation construction; by definition, then

$$H_n(\rho_M; \mathbb{Q})[M]_\mathbb{Q} = a \cdot [X] \qquad \text{and} \qquad H_n(\rho_{M'}; \mathbb{Q})[M']_\mathbb{Q} = a' \cdot [X],$$

where $n := \dim M = \dim M'$. *Assume* for a contradiction that there is a homotopy equivalence $f \colon M \longrightarrow M'$. By the universal property of rationalisation [7, Theorem 9.7(ii)] there is a continuous map $f_\mathbb{Q} \colon X \longrightarrow X$ with $\rho_{M'} \circ f = f_\mathbb{Q} \circ \rho_M$. Hence,

$$\begin{aligned} \deg_{[X]} f_\mathbb{Q} \cdot a \cdot [X] &= H_n(f_\mathbb{Q} \circ \rho_M; \mathbb{Q})[M]_\mathbb{Q} \\ &= H_n(\rho_{M'} \circ f; \mathbb{Q})[M]_\mathbb{Q} = \deg f \cdot a' \cdot [X]. \end{aligned}$$

Because f is a homotopy equivalence and because X is inflexible, it follows that $|\deg f| = 1 = |\deg_{[X]} f_\mathbb{Q}|$. Therefore, $|a| = |a'|$, which is a contradiction. So $M \not\simeq M'$. ☐

Example 9.9. Let $j \in \{1, \dots, 4\}$. In view of Remark 6.13, for all scalars $a \in \mathbb{Q} \setminus \{0\}$ the class $\mathcal{M}(A_j, a \cdot [A_j])$ is non-empty. Therefore, by the proposition above, there are infinitely many homotopy types of oriented closed simply connected manifolds having the rational homotopy type given by A_j; clearly, all of these manifolds are inflexible.

Similarly, for $j \in \{2,3,4\}$ and all $k \in \mathbb{N}_{>0}$ there are infinitely many homotopy types of oriented closed simply connected manifolds having the rational homotopy type given by $A_j^{\otimes k}$ (because the corresponding Witt index is trivial as well, and so also the scalar multiples of the fundamental class are realisable by manifolds).

For Propositions 9.10 and 9.13 below we shall need the follow lemma, which is a refinement of a special case of the Barge-Sullivan Theorem 6.11:

Lemma 9.10. *Let $(X, [X])$ be a $\mathbb{Q}$-Poincaré space of formal dimension $4k$ with vanishing Witt index: $\tau_{[X]} = 0 \in W_0(\mathbb{Q})$. Then $(X, [X])$ can be realised by a stably parallelisable oriented closed simply connected smooth manifold.*

Proof. The lemma follows from a little reflection upon the proof of the Barge-Sullivan theorem (Theorem 6.11). We need to find a stable bundle ξ over the rational space X such that the total Pontryagin class of ξ is trivial; hence we may choose ξ to be the trivial bundle. Since the manifold M produced by the Barge-Sullivan theorem has a normal map

$$\begin{array}{ccc} \nu_M & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{\bar{\nu}} & X \end{array}$$

where ν_M is the stable normal bundle of M, it follows that M is stably parallelisable. □

Corollary 9.11. *For each of the example dgas A_1, A_2, A_3 and A_4 of Section 8.1 and for each $a \in \mathbb{Q} \setminus \{0\}$, the class $\mathcal{M}(A_j, a \cdot [A_j])$ contains a stably parallelisable manifold.*

Proof. By Proposition 6.12 the $\mathbb{Q}$-Poincaré spaces $(X_j, a \cdot [X_j])$ realising the Poincaré dgas $(A_j, a \cdot [A_j])$ all have vanishing Witt index and so we may apply Lemma 9.10. □

In light of Theorem 9.5 we introduce some notation: for $j \in \{1, \dots, 4\}$ we write d_j for the formal dimension of A_j; more explicitly, $d_1 = 64$, $d_2 = 108$, $d_3 = 208$, $d_4 = 228$. Moreover, we abbreviate

$$\begin{aligned} D &:= \{d_1\} \cup \{d_j \cdot k \mid k \in \mathbb{N}_{>0}, j \in \{2,3,4\}\} \\ &= \{64\} \cup \{d \cdot k \mid k \in \mathbb{N}_{>0},\ d \in \{108, 208, 228\}\}. \end{aligned}$$

In dimensions in D we will now show that simply connected inflexible manifolds are "generic" from the point of view of rational bordism, thereby giving a first answer in the direction of Question 6.16.

Proposition 9.12 (Inflexible manifolds and rational bordism)**.** *Let $n \in D$. Then there is a positive integer $r(n)$, depending upon n, such that for any oriented closed n-manifold N the $r(n)$-fold disjoint union $\sqcup_{r(n)} N$, equivalently the r-fold connected sum $\#_{r(n)} N$, is oriented bordant to an oriented closed simply connected inflexible manifold.*

Proof. Because the products of complex projective spaces form a $\mathbb{Q}$-basis of the rational bordism ring $\Omega_*^{SO} \otimes \mathbb{Q}$ [18, Corollary 18.9] and because the torsion in Ω_*^{SO} has exponent 2 [25, Corollary 1] it suffices to consider the case where N is a product of complex projective spaces, say $N = \prod_{i=1}^m \mathbb{C}P^{n_i}$ with $2 \cdot (n_1 + \cdots + n_m) = n$.

By definition of D, we can write $n = d_j \cdot k$, with $j \in \{2,3,4\}$ and $k \in \mathbb{N}_{>0}$, or $j = 1 = k$. Moreover, let $M \in \mathcal{M}(A_j, [A_j])$; by Lemma 9.10 we may assume that M is stably parallelisable. Then $M^{\times k}$ is an oriented closed simply connected n-manifold that is inflexible (by Theorem 9.5) and stably parallelisable. In particular, $M^{\times k}$ is oriented null-bordant.

We now consider $N' := M^{\times k} \# N$. By construction, N' is oriented bordant to N and simply connected. It hence suffices to show that N' is inflexible: By Lemma 9.3, we have $\pi_{n-1}(M^{\times k}) \otimes \mathbb{Q} \cong \pi_{n-1}(M)^{\times k} \otimes \mathbb{Q} = 0$. By definition, $H^2(M^{\times k}; \mathbb{Q}) = 0$ if $j > 1$; in the case $n = d_1 = 64$, there is no class $x \in H^2(M; \mathbb{Q})$ with $x^{32} \neq 0$ (by definition, $H^2(M; \mathbb{Q}) \cong \mathbb{Q} \cdot x_1$, and $[x_1]^{32} = 0$, as shown in the proof of Proposition 8.5). However, there is a class $x \in H^2(N; \mathbb{Q}) = H^2(\prod_{i=1}^m \mathbb{C}P^{n_i})$ such that $x^{n/2}$ generates $H^n(N; \mathbb{Q})$. Therefore, $\deg(N, M^{\times k}) = \{0\}$, and now applying Theorem 9.1 shows that $N' = M^{\times k} \# N$ is inflexible. □

We saw above that there are many examples of stably parallelisable simply connected inflexible manifolds. On the other hand it is also possible to find simply connected inflexible manifolds with other tangential constraints. For example we have:

Proposition 9.13 (Non-spinable inflexible manifolds)**.** *For all $n \in D$ there are oriented closed simply connected non-spinable inflexible manifolds of dimension n.*

Proof. Let $N = S^{n-2} \tilde{\times} S^2$ be the total space of the sphere bundle of the non-trivial rank $(n-1)$-vector bundle over S^2. Then the second Stiefel-Whitney class of N generates $H^2(N; \mathbb{Z}/2) = \mathbb{Z}/2$ and N is non-spinable.

We write $n = d_j \cdot k$ with $j \in \{2,3,4\}$ and $k \in \mathbb{N}_{>0}$, or $j = 1 = k$. Then for all $M \in \mathcal{M}(A_j, [A_j])$ the manifold $M^{\times k}$ is inflexible (by Theorem 9.5) and simply connected. So $M^{\times k} \# N$ is non-spinable (because the Stiefel-Whitney class is non-trivial) and simply connected. We show now that $M^{\times k} \# N$ is inflexible:

As first step, we show that $\deg(N, M^{\times k}) = \{0\}$: A straightforward spectral sequence calculation shows that $H^d(N; \mathbb{Q}) = 0$ for all $d \in \{4,6,8,12\}$. On the other hand, by construction of the Poincaré dgas $A_1, \dots, A_4$ we have $H^d(M^{\times k}; \mathbb{Q}) \neq 0$ for some $d \in \{4,6,8,12\}$. Therefore, $\deg(N, M^{\times k}) = \{0\}$.

Furthermore, from Lemma 9.3 we obtain $\pi_{n-1}(M^{\times k}) \otimes \mathbb{Q} = 0$. Hence, $M^{\times k} \# N$ is inflexible by Theorem 9.1. □

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