COST VS. INTEGRAL FOLIATED SIMPLICIAL VOLUME

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Abstract. We show that integral foliated simplicial volume of closed manifolds gives an upper bound for the cost of the corresponding fundamental groups.

1. Introduction

The dynamical view on groups and spaces aims at understanding groups and topological spaces through actions on probability spaces. If Γ is a group and $\alpha = \Gamma \curvearrowright (X, \mu)$ is a measure preserving action on a probability space (X, μ), then one considers, for instance, the following invariants (see Section 2 and 3 for definitions and references): The cost $\operatorname{Cost}_\mu \alpha$ of α is a randomised version of the minimal number of generators of Γ. The cost $\operatorname{Cost} \Gamma$ of the group Γ is the infimum of all such $\operatorname{Cost}_\mu \alpha$.

If M is an oriented closed connected manifold with fundamental group Γ, then the α-parametrised simplicial volume $|M|^\alpha$ of M is a randomised version of the integral simplicial volume of M. The integral foliated simplicial volume $|M|$ of M is the infimum of all such $|M|^\alpha$.

In the residually finite case, the profinite completion provides a link between the dynamical view and the residually finite view: If Γ is a residually finite group, then the cost of the translation action of Γ on the profinite completion $\widehat{\Gamma}$ coincides with the rank gradient $\operatorname{rg} \Gamma$ of Γ (plus 1) [1, Theorem 1] and the corresponding parametrised simplicial volume of M coincides with the stable integral simplicial volume $\|M\|_{\mathbb{Z}}^\infty$ [15, Remark 6.7]. Moreover, these gradient invariants are related as follows:

Theorem 1.1 (rank gradient estimate [13, Theorem 1.1])**.** *If M is an oriented closed connected manifold with fundamental group Γ, then*

$$\operatorname{rg} \Gamma \leq \|M\|_{\mathbb{Z}}^\infty.$$

It is natural to wonder whether the corresponding dynamical estimate also holds [13, Question 1.3]. In the present article, we will complete the dynamical part of the picture by proving the following estimate:

Theorem 1.2 (cost estimate)**.** *Let M be an oriented closed connected manifold with fundamental group Γ and let $\alpha = \Gamma \curvearrowright (X, \mu)$ be an essentially free ergodic standard Γ-space. Then*

$$\operatorname{Cost}_\mu \alpha - 1 \leq |M|^\alpha.$$

In particular, $\operatorname{Cost} \Gamma - 1 \leq |M|$.

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Theorem 1.2 shows that integral foliated simplicial volume is a higher-dimensional, geometric refinement of the cost of groups. In particular, as in the case of the rank gradient estimate, the bound in Theorem 1.2 is far from being sharp in general: If M is an oriented closed connected hyperbolic surface, then [17, 10]

$$|\!|\!| M \times M |\!|\!| \geq \|M \times M\| \geq \|M\| \cdot \|M\| > 0,$$

but $\operatorname{Cost}(\pi_1(M) \times \pi_1(M)) = 1$ [11, Proposition 35.1].

The dependence of $\operatorname{Cost}_\mu \alpha$ and $|\!|\!| M |\!|\!|^\alpha$ on the chosen dynamical system α is a delicate open problem [8, 11][7, Section 1.5]. In analogy with the terminology for cost of groups, we define:

Definition 1.3 (cheap manifold, manifold of fixed price)**.** Let M be an oriented closed connected manifold.

- The manifold M is *cheap* if $|\!|\!| M |\!|\!| = 0$.
- The manifold M has *fixed price* if for all essentially free standard $\pi_1(M)$-spaces α and β we have $|\!|\!| M |\!|\!|^\alpha = |\!|\!| M |\!|\!|^\beta$.

As for groups, it is not known whether all manifolds have fixed price.

Corollary 1.4. *Let M be an oriented closed connected manifold with fundamental group Γ.*

(1) *If M is cheap, then Γ is cheap.*
(2) *If M is cheap and of fixed price, then Γ is cheap and of fixed price.*

Proof. Let M be cheap. We first observe that this implies that Γ is infinite (if Γ is finite, then $|\!|\!| M |\!|\!| = 1/|\Gamma| \cdot \|\widetilde{M}\|_{\mathbb{Z}}$ [15, Corollary 6.3], which is non-zero). Because Γ is infinite, $\operatorname{Cost}\Gamma \geq 1$ [11, p. 108]. On the other hand, Theorem 1.2 yields that $\operatorname{Cost}\Gamma \leq |\!|\!| M |\!|\!| + 1 = 1$; therefore, $\operatorname{Cost}\Gamma = 1$, which means that Γ is cheap.

Let now M additionally have fixed price. In view of ergodic decomposition [11, Corollary 18.6], it suffices to show that $\operatorname{Cost}_\mu \alpha = 1$ holds for all essentially free *ergodic* standard Γ-spaces $\alpha = \Gamma \curvearrowright (X, \mu)$. In this case, again Theorem 1.2 shows that

$$1 \leq \operatorname{Cost}\Gamma \leq \operatorname{Cost}_\mu \alpha \leq |\!|\!| M |\!|\!|^\alpha + 1 = 1. \qquad \square$$

The class of cheap manifolds of fixed price is known to include all oriented closed connected manifolds that

- are aspherical and have infinite amenable fundamental group [7, Theorem 1.9],
- are smooth and admit a smooth S^1-action without fixed points and whose orbits are π_1-injective [3, Theorem 1.1] or that are smooth and aspherical and admit a smooth non-trivial S^1-action [3, Corollary 1.2],
- are generalised graph manifolds [4, Theorem 1.6],
- are a product of a cheap manifold of fixed price and another manifold [17, proof of Theorem 5.34],
- are smooth, aspherical, and have trivial minimal volume [2, (proof of) Corollary 5.4].

For the sake of completeness, we put Theorem 1.2 in context with L^2-Betti numbers: For L^2-Betti numbers, we a have a harmonic correspondence between the classical, the dynamical, and the residually finite view: L^2-Betti numbers of compact manifolds can be described both in the residually finite view (as Betti number gradients) [16] and in the dynamical view (as L^2-Betti numbers of orbit relations) [9]. In contrast, it is known that stable integral simplicial volume and integral foliated simplicial volume, in general do *not* coincide with the classicial simplicial volume of aspherical oriented closed connected manifolds [6, Theorem 2.1][7, Theorem 1.8]. Integral foliated simplicial volumes and L^2-Betti numbers of an oriented closed connected manifold M are for every $k \in \mathbb{N}$ linked by the following chain of inequalities [17, Corollary 5.28 (the constant factor can be improved to 1)][15, Proposition 6.1]:

$$b_k^{(2)}(M) \leq \mathopen{\pmb|} M \mathclose{\pmb|} \leq \|M\|_{\mathbb{Z}}^{\infty}.$$

Moreover, it is known that $b_1^{(2)}(\pi_1(M)) \leq \operatorname{Cost} \pi_1(M) - 1$ (if $\pi_1(M)$ is infinite) [9, Corollaire 3.23]. Hence, Theorem 1.2 is a refinement of this chain of inequalities in degree 1:

$$b_1^{(2)}(M) \leq \operatorname{Cost} \pi_1(M) - 1 \leq \mathopen{\pmb|} M \mathclose{\pmb|} \leq \|M\|_{\mathbb{Z}}^{\infty}.$$

However, the following problem remains open:

Question 1.5. Let M be an oriented closed connected aspherical manifold whose simplicial volume $\|M\|$ satisfies $\|M\| = 0$. Does this already imply that $\pi_1(M)$ is cheap?

Remark 1.6. If the Singer conjecture for L^2-Betti numbers is true and the conjecture that $b_1^{(2)}(\Gamma) = \operatorname{Cost} \Gamma - 1$ holds for every (finitely presented infinite) group is true, then Question 1.5 clearly has a positive answer (even independently of the simplicial volume in dimension at least 3). However, as these two conjectures seem to be wild and wide open, it would be interesting to find an alternative, direct, answer to Question 1.5.

Organisation of this article. We first review the notion of cost of standard equivalence relations (Section 2) and establish a basic estimate for cost of certain subrelations (Section 2.4). We then recall the notion of integral foliated simplicial volume (Section 3). In Section 4, we will prove Theorem 1.2. Finally, in Section 4.6, we will discuss the weightless version of Theorem 1.2.

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2. Cost

The cost of a dynamical system of a group is a randomised version of the rank (i.e., minimal number of generators) of the group. More generally, one can consider the cost of standard Borel equivalence relations on measure spaces. More information about these subjects can be found in the literature [8, 9, 11].

2.1. **Standard actions and equivalence relations.** We will use the following notation and conventions on standard equivalence relations:

A *standard Borel measure [probability] space* is a measure space [probability space] (X, μ), where the measurable space X is isomorphic to a Polish space with its Borel σ-algebra. For simplicity, we will only consider the case of standard Borel measure spaces with finite total measure.

A *measurable equivalence relation* on a standard Borel (measure) space X is a measurable subset $S \subset X \times X$ that is an equivalence relation on X. The *automorphism group of* S (or *full group of* S) is the group $[S]$ (via composition) of measurable isomorphisms $f \colon X \longrightarrow X$ that satisfy

$$\forall_{x,y \in X} \quad x \sim_S y \Longrightarrow f(x) \sim_S f(y).$$

Moreover, $[\![S]\!]$ denotes the set of partial automorphisms of S, i.e., of measurable isomorphisms $f \colon A \longrightarrow B$ between measurable subsets $A, B \subset X$ that satisfy

$$\forall_{x,y \in A} \quad x \sim_S y \Longrightarrow f(x) \sim_S f(y);$$

we write $\operatorname{dom} f := A$ for the *domain* of f.

Definition 2.1 (standard equivalence relation)**.** A *standard equivalence relation* on a standard Borel measure space (X, μ) is a measurable equivalence relation S on X, where each equivalence class has cardinality at most $|\mathbb{N}|$ and where each element of $[S]$ is μ-preserving.

One of the key objects of measurable group theory and the dynamical view is the orbit relation of a group action:

Example 2.2 (orbit relation of an action)**.** Let Γ be a group. A *standard* Γ-*space* is a standard Borel probability space (X, μ) together with a measurable μ-preserving (left) action of Γ on (X, μ).

If Γ is countable and $\alpha = \Gamma \curvearrowright (X, \mu)$ is a standard Γ-space, then the *orbit relation*

$$\bigl\{(x, \gamma \cdot x) \bigm| x \in X,\ \gamma \in \Gamma\bigr\} \subset X \times X$$

is a standard equivalence relation in the sense above.

Conversely, it can be shown that every standard equivalence relation arises as orbit relation of a suitable action of a suitable countable group on the underlying standard Borel measure space [5, Theorem 1].

Moreover, we will need the following terms and constructions: Let S be a standard equivalence relation on a standard Borel measure space (X, μ).

- If $A \subset X$ is a measurable subset with $\mu(A) > 0$, then the restriction $\mu|_A$ of μ to A turns $(A, \mu|_A)$ into a standard Borel measure space.
- In this situation, the restriction
$$S|_A := \bigl\{(x, y) \in A \times A \bigm| x \sim_S y\bigr\}$$
of S to A is a standard equivalence relation.
- A measurable subset $A \subset X$ is a *[almost] complete section of* S, if for [μ-almost] every $x \in X$ there is a $y \in A$ with $x \sim_S y$.
- The relation S on X is *aperiodic*, if for μ-almost every $x \in X$ the S-orbit $S \cdot x := \{y \in X \mid y \sim_S x\}$ is infinite.

- A measurable subset $A \subset X$ is *S-invariant* if

$$\forall_{x \in A} \quad S \cdot x \subset A.$$

- The relation S on (X, μ) is *ergodic*, if every S-invariant measurable subset A of X satisfies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

2.2. **Cost of a standard equivalence relation.** The cost of a standard equivalence relation is the minimal "number" of "generators" needed to describe the relation:

Definition 2.3 (graphing, cost)**.** Let S be a standard equivalence relation on a standard Borel measure space (X, μ).

- Let $\Phi = (\varphi_i)_{i \in I}$ be a family of elements of $[\![S]\!]$. Then

$$\langle \Phi \rangle_X := \Bigl\langle \bigcup_{i \in I} \bigl\{ (x, \varphi_i(x)) \in X \times X \mid x \in \operatorname{dom} \varphi_i \bigr\} \Bigr\rangle_X$$

  denotes the smallest (with respect to inclusion) equivalence relation on X containing the given set of pairs.
- A *graphing of S* is a family $\Phi = (\varphi_i)_{i \in I}$ in $[\![S]\!]$ with $\langle \Phi \rangle_X = S$. The *cost* of Φ is defined as

$$\operatorname{Cost}_\mu \Phi := \sum_{i \in I} \mu(\operatorname{dom} \varphi_i).$$

- The *cost* $\operatorname{Cost}_\mu S$ of S is the infimum of all costs of graphings of S.

Definition 2.4 (cost of a group [8])**.** The *cost* $\operatorname{Cost} \Gamma$ of a countable group Γ is the infimum of all costs of orbit relations of standard Γ-spaces.

Example 2.5. Let Γ be a finitely generated group. Then $\operatorname{Cost} \Gamma \leq \operatorname{rk} \Gamma$, where $\operatorname{rk} \Gamma$ denotes the minimal number of generators of Γ (as witnessed by the translation automorphisms associated with a smallest generating set). If Γ in addition is residually finite and infinite, then the translation action of Γ on its profinite completion $\widehat{\Gamma}$ is a standard Γ-space and [1, Theorem 1]

$$\operatorname{Cost}_\mu(\Gamma \curvearrowright \widehat{\Gamma}) - 1 = \operatorname{rg} \Gamma$$

(where $\operatorname{rg} \Gamma$ denotes the rank gradient of Γ).

In all cases, where the cost $\operatorname{Cost} \Gamma$ of a countable infinite group Γ could be computed so far, it coincides with $b_1^{(2)}(\Gamma) + 1$. This includes, for example, amenable groups, free groups, etc. [8, 9, 11].

2.3. **Cost and restrictions.** We collect basic facts on cost with respect to restrictions.

Lemma 2.6 (cost of partitions [11, p. 60])**.** *Let (X, μ) be a standard Borel measure space, let R be a standard equivalence relation on X, and let $X = \bigcup_{j=1}^m A_j$ be a partition of X into measurable R-invariant subsets of non-zero measure. Then*

$$\operatorname{Cost}_\mu R = \sum_{j=1}^m \operatorname{Cost}_{\mu|_{A_j}} R|_{A_j}.$$

Lemma 2.7 (cost of complete sections [8, Proposition II.6][11, Proposition 21.1])**.** *Let (X, μ) be a standard Borel measure space, let R be a standard equivalence relation on X, and let $A \subset X$ be a complete section of R. Then*

$$\operatorname{Cost}_\mu R = \operatorname{Cost}_{\mu|_A}(R|_A) + \mu(X \setminus A).$$

Lemma 2.8 (cost of restrictions)**.** *Let (X, μ) be a standard Borel probability space, let R be a standard equivalence relation on X, and let $A \subset X$ be a measurable subset with $\mu(A) > 0$. Then*

$$\operatorname{Cost}_{\mu|_A}(R|_A) \leq \operatorname{Cost}_\mu R.$$

Proof. Let $B := R \cdot A = \bigcup_{x \in A} R \cdot x \subset X$. Then B is a measurable subset of X [5, p. 291]. By construction, B and $X \setminus B$ are measurable R-invariant subsets of X. If $\mu(X \setminus B) = 0$, then A is an (almost) complete section of R and Lemma 2.7 shows that

$$\operatorname{Cost}_{\mu|_A}(R|_A) = \operatorname{Cost}_\mu R - \mu(X \setminus A) \leq \operatorname{Cost}_\mu R.$$

If $\mu(X \setminus B) \neq 0$, then we can apply Lemma 2.6 and Lemma 2.7 (because A is a complete section of $R|_B$ on B) to obtain

$$\begin{aligned} \operatorname{Cost}_\mu R &= \operatorname{Cost}_{\mu|_B}(R|_B) + \operatorname{Cost}_{\mu|_{X \setminus B}}(R|_{X \setminus B}) \\ &\geq \operatorname{Cost}_{\mu|_A}(R|_A) + \mu(B \setminus A) \\ &\geq \operatorname{Cost}_{\mu|_A}(R|_A). \end{aligned}$$

□

2.4. **Cost of translation finite extensions.** The key estimate in the proof of Theorem 1.2 will involve the following variation of the notion of finite index subrelations. In contrast with finite index subrelations, we only require that the orbits of the ambient relation can be covered, in a uniform way, by finitely many translates of orbits of the subrelation:

Definition 2.9 (translation finite extension)**.** Let (X, μ) be a standard Borel probability space, let S be a standard equivalence relation on (X, μ), and let $R \subset S$ be a standard equivalence relation on (X, μ) that is contained in S. Then $R \subset S$ is a *translation finite extension* if there exists a finite set $F \subset [S]$ such that for μ-almost every $x \in X$ we have

$$S \cdot x = \bigcup_{f,g \in F} f\bigl(R \cdot g^{-1}(x)\bigr).$$

Example 2.10. Let (X, μ) be a standard Borel probability space.

- Let $\Gamma \curvearrowright (X, \mu)$ be a standard Γ-space and let S be the corresponding orbit relation on X. Moreover, let $\Lambda \subset \Gamma$ be a finite index subgroup and let $R \subset S$ be the orbit relation of the action restricted to Λ. Then $R \subset S$ is a translation finite extension (witnessed by the left translations of a set of coset representatives).

  In this case, R is even a subrelation of finite index of S and thus we have $\operatorname{Cost}_\mu S \leq \operatorname{Cost}_\mu R$ [8, Proposition VI.23][11, Proposition 25.1].
- Conversely, if $R \subset S$ is a translation finite extension of standard equivalence relations on (X, μ), then R does not necessarily have finite index in S: We consider the circle $S^1 = [0, 1]/(0 \sim 1)$ with the Lebesgue probability measure μ and a $\mathbb{Z}$-action by irrational

rotation. Let S be the corresponding orbit relation. Let $\pi\colon [0,1] \longrightarrow S^1$ be the canonical projection, let $A := \pi([0,1/2])$, and let

$$R := \langle S|_A\rangle_X \subset S.$$

Then R does *not* have finite index in S, but $R \subset S$ is a translation finite extension.

Moreover, $\operatorname{Cost}_\mu(S) = 1$ [8, Corollaire III.4][11, Corollary 31.2] and (Lemma 2.6 and Lemma 2.7)

$$\begin{aligned}\operatorname{Cost}_\mu(R) &= \operatorname{Cost}_{\mu|_A}(R|_A) + \operatorname{Cost}_{\mu|_{S^1\setminus A}}(R|_{S^1\setminus A})\\ &= \operatorname{Cost}_{\mu|_A}(S|_A) + 0\\ &= \operatorname{Cost}_\mu(S) - \mu(S^1 \setminus A)\\ &= 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

In particular, in this case we have $\operatorname{Cost}_\mu S \not\leq \operatorname{Cost}_\mu R$.

Lemma 2.11 (cost estimate for translation-finite extensions)**.** *Let (X,μ) be a standard Borel probability space, let S be an aperiodic ergodic standard relation on (X,μ), and let $R \subset S$ be a translation finite extension. Then*

$$\operatorname{Cost}_\mu S \leq \operatorname{Cost}_\mu R + 1.$$

Proof. The proof is a straightforward adaption of the (stronger) cost estimate for finite index subrelations [11, Proposition 25.1]: Because $R \subset S$ is translation finite extension, there exists a finite set $F \subset [S]$ such that for μ-almost every $x \in X$ we have

$$S \cdot x = \bigcup_{f,g\in F} f\bigl(R \cdot g^{-1}(x)\bigr).$$

Passing to an S-invariant co-null subset, we may assume without loss of generality that this even holds for every $x \in X$. Let $\varepsilon \in \mathbb{R}_{>0}$ and let Φ be a graphing of R with

$$\operatorname{Cost}_\mu(\Phi) \leq \operatorname{Cost}_\mu R + \varepsilon.$$

Furthermore, let $A \subset X$ be a complete Borel section of S with $0 < \mu(A) < \varepsilon$ (such a set does exist [11, Lemma 6.7]). Thus, by Lemma 2.7,

$$\operatorname{Cost}_\mu(S) = \operatorname{Cost}_\mu(S|_A) + \mu(X \setminus A) \leq \operatorname{Cost}_{\mu|_A}(S|_A) + 1.$$

For $f \in F$, we let

$$\begin{aligned}\varphi_f := f^{-1}|_A \colon A &\longrightarrow f^{-1}(A)\\ x &\longmapsto f^{-1}(x)\end{aligned}$$

and $\Phi_A := \Phi \cup (\varphi_f)_{f\in F}$; finally, we set

$$\overline{R} := \langle \Phi_A\rangle_X.$$

Then $\overline{R}|_A = S|_A$, as the following calculation shows: Let $x \in A$. By construction, $\varphi_f \in [\![S]\!]$ for every $f \in F$. In particular, $\overline{R} \subset S$ and thus $\overline{R}|_A \subset S|_A$. Conversely, let $x,y \in A$ with $x \sim_S y$. Then there exist $f,g \in F$ with $y \in f(R\cdot g^{-1}(x))$, whence $f^{-1}(y) \sim_R g^{-1}(x)$. By construction, we thus have

$$y \sim_{\overline{R}} f^{-1}(y) \quad\wedge\quad f^{-1}(y) \sim_{\overline{R}} g^{-1}(x) \quad\wedge\quad g^{-1}(x) \sim_{\overline{R}} x,$$

and so $y \sim_{\overline{R}} x$. This shows $\overline{R}|_A = S|_A$.

In combination with Lemma 2.8 we obtain

$$\begin{aligned}\operatorname{Cost}_{\mu|_A}(S|_A) &= \operatorname{Cost}_{\mu|_A}(\overline{R}|_A) \leq \operatorname{Cost}_\mu(\overline{R}) \\ &\leq \operatorname{Cost}_\mu(\Phi_A) \leq \operatorname{Cost}_\mu(\Phi) + |F| \cdot \mu(A) \\ &\leq \operatorname{Cost}_\mu(R) + \varepsilon + |F| \cdot \varepsilon.\end{aligned}$$

Taking $\varepsilon \to 0$ shows that

$$\operatorname{Cost}_\mu(S) \leq \operatorname{Cost}_{\mu|_A}(S|_A) + 1 \leq \operatorname{Cost}_\mu(R) + 1,$$

as claimed. □

3. Integral foliated simplicial volume

Simplicial volumes are defined as the minimal number (measured in a suitable sense) of singular simplices needed to build the given manifold [10, 12]. In the case of integral foliated simplicial volume, we use bounded functions on dynamical systems of the fundamental group as coefficients. More information and computations can be found in the literature [17, 15, 7, 3, 4, 2].

Let M be an oriented closed connected n-manifold with fundamental group Γ and let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a standard Γ-space. Then $L^\infty((X,\mu),\mathbb{Z})$ inherits a right $\mathbb{Z}\Gamma$-module structure and we write

$$C_*(M;\alpha) := L^\infty((X,\mu),\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z})$$

for the corresponding chain complex with twisted coefficients.

A chain $c \in C_*(M;\alpha)$ is an *α-parametrised fundamental cycle* if it is homologous (in the complex $C_*(M;\alpha)$) to the image of a $\mathbb{Z}$-fundamental cycle on M under the canonical inclusion $C_*(M;\mathbb{Z}) \longrightarrow C_*(M;\alpha)$. If $c = \sum_{j=1}^m f_j \otimes \sigma_j \in C_n(M;\alpha)$ is in reduced form (i.e., all $\sigma_1, \dots, \sigma_m$ lie in different Γ-orbits under the deck transformation action), then

$$|c|_1 := \sum_{j=1}^m \int_X |f_j| \, d\mu \in \mathbb{R}_{\geq 0}.$$

Definition 3.1 (parametrised simplicial volume, integral foliated simplicial volume)**.** Let M be an oriented closed connected n-manifold.

- The *α-parametrised simplicial volume* of M is defined by
$$\mathopen{|\!}M\mathclose{|\!}^\alpha := \inf\bigl\{|c|_1 \bigm| c \in C_n(M;\alpha) \text{ is an } \alpha\text{-parametrised fundamental cycle of } M\bigr\}.$$
- The *integral foliated simplicial volume* $\mathopen{|\!}M\mathclose{|\!}$ of M is the infimum of all parametrised simplicial volumes of M.

If M is an oriented closed connected manifold, then classical simplicial volume, integral foliated simplicial volume, and stable integral simplicial volume are related by the chain [17, Theorem 5.35][15, Proposition 6.1]

$$\|M\| \leq \mathopen{|\!}M\mathclose{|\!} \leq \|M\|_{\mathbb{Z}}^\infty.$$

Example 3.2. Let M be an oriented closed connected manifold with residually finite fundamental group Γ. Then $\mathopen{|\!}M\mathclose{|\!}^{\widehat{\Gamma}}$ coincides with the stable integral simplicial volume of M [15, Remark 6.7].

4. Proof of Theorem 1.2

We will now prove Theorem 1.2.

4.1. **The case of finite fundamental group.** Let us first get the (pathological) case that the fundamental group Γ is finite out of the way: If $\alpha = \Gamma \curvearrowright (X, \mu)$ is a standard Γ-space, then [8, Corollaire I.10][11, Proposition 22.1]

$$\operatorname{Cost}_\mu \alpha - 1 \leq 1 - 1 = 0 \leq |M|^\alpha.$$

Taking the infimum over all such α shows that $\operatorname{Cost} \Gamma - 1 \leq |M|$. This proves Theorem 1.2 if Γ is finite.

4.2. **Setup.** In view of Section 4.1, we will assume for the rest of the proof of Theorem 1.2 that the fundamental group Γ of M is infinite. Moreover, we will fix the following notation:

- Let $\alpha = \Gamma \curvearrowright (X, \mu)$ be an essentially free ergodic standard Γ-space and let $S \subset X \times X$ be the corresponding orbit relation. Hence, S is aperiodic and ergodic.
- Let $D \subset \widetilde{M}$ be a set-theoretic, relatively compact, fundamental domain for the deck transformation action of Γ on $\widetilde{M}$.
- Let

$$c = \sum_{j=1}^m f_j \otimes \sigma_j \in C_n(M; \alpha) = L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M}; \mathbb{Z})$$

  with $f_1, \dots, f_m \in L^\infty(X, \mathbb{Z})$, $\sigma_1, \dots, \sigma_m \in \operatorname{map}(\Delta^n, \widetilde{M})$ be an α-parametrised fundamental cycle of M. Moreover, we may assume that the representation of c is in *reduced form*, i.e., that $\sigma_j(v_0) \in D$ for all $j \in \{1, \dots, m\}$ and that the singular simplices $\sigma_1, \dots, \sigma_m$ are all different.
- For $j \in \{1, \dots, m\}$ let $\gamma_j \in \Gamma$ be the unique group element satisfying $\sigma_j(v_1) \in \gamma_j \cdot D$. We then consider $\varphi_j \in [\![S]\!]$ given by

$$\begin{aligned} \varphi_j \colon A_j &\longrightarrow \gamma_j^{-1} \cdot A_j \\ x &\longmapsto \gamma_j^{-1} \cdot x, \end{aligned}$$

  where $A_j := \operatorname{supp} f_j \subset X$. Let $R := \langle \varphi_1, \dots, \varphi_m \rangle_X$ be the standard equivalence relation on X generated by $\varphi_1, \dots, \varphi_m$.

By construction, the cost of the relation R is controlled in terms of $|c|_1$:

Lemma 4.1. *In the situation of Setup 4.2, the relation R is a subrelation of S and*

$$\operatorname{Cost}_\mu R \leq \sum_{j=1}^m \mu(A_j) \leq |c|_1.$$

Proof. By construction, $\Phi := (\varphi_j)_{j \in \{1,\dots,m\}}$ is a graphing of R and we have $\varphi_1, \dots, \varphi_m \in [\![S]\!]$. Therefore, $R \subset S$ and

$$\operatorname{Cost}_\mu R \leq \operatorname{Cost}_\mu \Phi = \sum_{j=1}^m \mu(A_j).$$

Moreover, $\sum_{j=1}^m \mu(A_j) \leq \sum_{j=1}^m \int_X |f_j| d\mu = |c|_1$, because each f_j is integer-valued and the representation $\sum_{j=1}^m f_j \otimes \sigma_j$ of c is in reduced form. □

Remark 4.2. Integration $L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; \mathbb{Z}) \longrightarrow C_*(M; \mathbb{R})$ of the coefficients and a covering theoretic argument show that $\langle \gamma_1, \dots, \gamma_m \rangle_\Gamma$ is a finite index subgroup of Γ. However, in general, the subrelation R of S will *not* have finite index in S (this can already be seen in the case of Schmidt's parametrised fundamental cycles of S^1 [17, proof of Proposition 5.30]).

In view of Lemma 4.1 it suffices to prove that $\operatorname{Cost}_\mu S - 1 \leq \operatorname{Cost}_\mu R$. To this end, we will establish that $R \subset S$ is a translation finite extension and then apply Lemma 2.11.

4.3. **Passing to locally finite chains.** In order to prove that $R \subset S$ is a translation finite extension, it is convenient to pass to locally finite chains.

Remark 4.3 (locally finite cycles)**.** In the situation of Setup 4.2, for μ-almost every $x \in X$, the chain

$$c_x := \sum_{j=1}^m \sum_{\gamma \in \Gamma} f_j(\gamma^{-1} \cdot x) \cdot \gamma \cdot \sigma_j \in C_n^{\mathrm{lf}}(\widetilde{M}; \mathbb{Z})$$

given by evaluation on the Γ-orbit of x is a well-defined locally finite fundamental cycle of $\widetilde{M}$ [7, Lemma 2.5].

We therefore recall a basic property of locally finite chains.

Lemma 4.4. *Let N be an oriented connected n-manifold, let Z be a commutative ring with unit, and let $x \in N$. Then the restriction map induces a well-defined isomorphism*

$$\begin{aligned} \varrho_x \colon H_n^{\mathrm{lf}}(N; Z) &\longrightarrow H_n(N, N \setminus \{x\}; Z) \\ \Bigl[\sum_{j \in J} a_j \cdot \sigma_j\Bigr] &\longmapsto \Bigl[\sum_{j \in J, x \in \sigma_j(\Delta^n)} a_j \cdot \sigma_j\Bigr]. \end{aligned}$$

In particular: If $c = \sum_{j \in J} a_j \cdot \sigma_j \in C_n^{\mathrm{lf}}(N; Z)$ is a locally finite cycle whose associated class $[c] \in H_n^{\mathrm{lf}}(N; Z)$ is non-trivial, then there exists $j \in J$ such that

$$x \in \sigma_j(\Delta^n).$$

Proof. The restriction map on the chain level extends to a well-defined chain map $C_*^{\mathrm{lf}}(N; Z) \longrightarrow C_*(N, N \setminus \{x\}; Z)$. Checking the effect of ϱ_x on the locally finite fundamental class of N proves the first claim.

The second part is a direct consequence of the first part. □

4.4. **Establishing translation finiteness.**

Lemma 4.5. *In the situation of Setup 4.2, $R \subset S$ is a translation finite extension in the sense of Definition 2.9.*

Proof. For the proof we will use geometric properties of c (and its locally finite companions) on $\widetilde{M}$. Let

$$K := D \cup \bigcup_{j=1}^m \sigma_j(\Delta^n) \subset \widetilde{M}.$$

Then D and K are relatively compact and hence

$$F := \{f \in \Gamma \mid D \cap f \cdot K \neq \emptyset\}$$

is finite. We will now show that μ-almost every S-orbit is covered by the F-translates of orbits of R:

Let $x \in X$ be such that the evaluation c_x is a locally finite $\mathbb{Z}$-fundamental cycle of $\widetilde{M}$ (Remark 4.3). We associate the following graph $G_x = (V_x, E_x)$ with c_x:

- vertices: we set $V_x := \bigl\{\gamma \in \Gamma \bigm| \exists_{j\in\{1,\dots,m\}} \quad \gamma^{-1} \cdot x \in A_j\bigr\} \subset \Gamma$.
- edges: we set

$$\begin{aligned} E_x := \bigl\{\{\gamma, \lambda\} \bigm| \gamma, \lambda \in V_x \wedge \gamma \neq \lambda \\ \wedge \exists_{i,j\in\{1,\dots,m\}} \quad \exists_{k,\ell\in\{0,\dots,n\}} \\ \bigl(\, \partial_k(\gamma \cdot \sigma_i) = \partial_\ell(\lambda \cdot \sigma_j) \\ \wedge \gamma^{-1} \cdot x \in A_i \wedge \lambda^{-1} \cdot x \in A_j\bigr)\bigr\}. \end{aligned}$$

The combinatorics of G_x will allow us to link the orbits of R with the geometry of c_x. More precisely, we will establish the following facts:

(1) For all $\{\gamma, \lambda\} \in E_x$, we have $\gamma^{-1} \cdot x \sim_R \lambda^{-1} \cdot x$.

(2) If $V \subset V_x$ is (the set of vertices of) a connected component of G_x, then

$$c_{x,V} := \sum_{j=1}^{m} \sum_{\gamma\in V} f_j(\gamma^{-1} \cdot x) \cdot \gamma \cdot \sigma_j$$

is a well-defined cycle in $C_n^{\mathrm{lf}}(\widetilde{M}; \mathbb{Z})$.

(3) Let $x_0 \in D$. There exists a connected component $V \subset V_x$ of G_x such that $[c_{x,V}] \neq 0$ in $H_n^{\mathrm{lf}}(\widetilde{M}; \mathbb{Z})$ and

$$\exists_{g\in V} \quad \exists_{j\in\{1,\dots,m\}} \quad x_0 \in g \cdot \sigma_j(\Delta^n) \subset g \cdot K.$$

Hence, any such g is in F.

(4) We have $F \cdot V^{-1} = \Gamma$, where $V^{-1} := \{\gamma^{-1} \mid \gamma \in V\}$.

Proof of (1). By definition of E_x, there are $i, j \in \{1, \dots, m\}$ and $k, \ell \in \{0, \dots, n\}$ with

$$\gamma^{-1} \cdot x \in A_i \;\wedge\; \lambda^{-1} \cdot x \in A_j \;\wedge\; \partial_k(\gamma \cdot \sigma_i) = \partial_\ell(\lambda \cdot \sigma_j).$$

We now distinguish the following cases:

- If $k > 0$ and $\ell > 0$, then

$$\gamma \cdot \sigma_i(v_0) = \bigl(\partial_k(\gamma \cdot \sigma_i)\bigr)(v_0) = \bigl(\partial_\ell(\lambda \cdot \sigma_j)\bigr)(v_0) = \lambda \cdot \sigma_j(v_0).$$

  In particular, $\gamma \cdot D \cap \lambda \cdot D \neq \emptyset$, and so $\gamma = \lambda$ (whence $\gamma^{-1} \cdot x \sim_R \lambda^{-1} \cdot x$).
- If $k = 0$ and $\ell > 0$, then

$$\gamma \cdot \sigma_i(v_1) = \bigl(\partial_k(\gamma \cdot \sigma_i)\bigr)(v_0) = \bigl(\partial_\ell(\lambda \cdot \sigma_j)\bigr)(v_0) = \lambda \cdot \sigma_j(v_0).$$

  By definition of γ_i, we have $\sigma_i(v_1) \in \gamma_i \cdot D$. Therefore, we obtain that $\gamma \cdot \gamma_i \cdot D \cap \lambda \cdot D \neq \emptyset$, and thus $\gamma \cdot \gamma_i = \lambda$. Because of $\gamma^{-1} \cdot x \in A_i$, the definition of R shows that

$$\lambda^{-1} \cdot x = \gamma_i^{-1} \cdot \gamma^{-1} \cdot x \sim_R \gamma^{-1} \cdot x.$$

- If $k > 0$ and $\ell = 0$, we can argue as in the previous case.

- If $k = 0$ and $\ell = 0$, then
$$\gamma \cdot \sigma_i(v_1) = \bigl(\partial_k(\gamma \cdot \sigma_i)\bigr)(v_0) = \bigl(\partial_\ell(\lambda \cdot \sigma_j)\bigr)(v_0) = \lambda \cdot \sigma_j(v_1).$$
Similarly, to the previous cases, we obtain $\gamma \cdot \gamma_i = \lambda \cdot \gamma_j$. Hence
$$\lambda^{-1} \cdot x \sim_R \gamma_j^{-1} \cdot \lambda^{-1} \cdot x = \gamma_i^{-1} \cdot \gamma^{-1} \cdot x$$
(via φ_j) and
$$\gamma_i^{-1} \cdot \gamma^{-1} \cdot x \sim_R \gamma^{-1} \cdot x$$
(via φ_i). By transitivity, it follows that $\lambda^{-1} \cdot x \sim_R \gamma^{-1} \cdot x$.

Proof of (2). Let $\pi_0(G_x)$ be the set (of vertex sets) of the connected components of G_x. The sum decomposition $c_x = \sum_{V \in \pi_0(G_x)} c_{x,V}$ is a locally finite sum of locally finite chains. Hence,
$$\begin{aligned} 0 = \partial(c_x) &= \sum_{V \in \pi_0(G_x)} \partial(c_{x,V}) \\ &= \sum_{V \in \pi_0(G_x)} \sum_{j=1}^{m} \sum_{\gamma \in V} \sum_{k=0}^{n} (-1)^k \cdot f_j(\gamma^{-1} \cdot x) \cdot \partial_k(\gamma \cdot \sigma_j). \end{aligned}$$
By construction of the graph G_x, if $V, W \in \pi_0(G_x)$ are different components, then the terms of $\partial(c_{x,V})$ and $\partial(c_{x,W})$ cannot interfere with each other. Therefore, we obtain
$$\partial(c_{x,V}) = 0$$
for all $V \in \pi_0(G_x)$.

Proof of (3). By (2), $c_x = \sum_{V \in \pi_0(G_x)} c_{x,V}$ is a locally finite sum of cycles. Applying the restriction homomorphism ϱ_{x_0} of Lemma 4.4 gives the effectively finite decomposition
$$0 \neq [\widetilde{M}, \widetilde{M} \setminus \{x_0\}]_{\mathbb{Z}} = \varrho_{x_0}[\widetilde{M}]^{\mathrm{lf}}_{\mathbb{Z}} = \varrho_{x_0}[c_x] = \sum_{V \in \pi_0(G_x)} \varrho_{x_0}[c_{x,V}].$$
Hence, there exists a connected component $V \in \pi_0(G_x)$ with $[c_{x,V}] \neq 0$ in $H_n^{\mathrm{lf}}(\widetilde{M}; \mathbb{Z})$. By Lemma 4.4, there exist $g \in V$ and $j \in \{1, \dots, m\}$ with
$$x_0 \in g \cdot \sigma_j(\Delta^n).$$
By definition of F and because $x_0 \in D$, this implies $g \in F$.

Proof of (4). Clearly, $F \cdot V^{-1} \subset \Gamma$. Conversely, let $\gamma \in \Gamma$. Applying Lemma 4.4 to the point $\gamma^{-1} \cdot x_0$ and the class $[c_{x,V}] \in H_n^{\mathrm{lf}}(\widetilde{M}; \mathbb{Z})$ yields that there exists a $\lambda \in V$ and $j \in \{1, \dots, m\}$ with
$$\gamma^{-1} \cdot x_0 \in \lambda \cdot \sigma_j(\Delta^n).$$
Thus, $\gamma^{-1} \cdot D \cap \lambda \cdot K \neq \emptyset$, and so $\gamma \cdot \lambda \in F$. Hence, $\gamma \in F \cdot \lambda^{-1} \subset F \cdot V^{-1}$.

Conclusion of proof: Let $V \subset V_x$ and $g \in V \cap F$ be as provided by fact (3). Then (1) shows that
$$V^{-1} \cdot x \subset R \cdot g^{-1} \cdot x.$$
Using (4), we obtain that
$$S \cdot x = \Gamma \cdot x = \bigcup_{f \in F} f \cdot V^{-1} \cdot x \subset \bigcup_{f,g \in F} f \cdot R \cdot g^{-1}(x) \subset S \cdot x.$$
Because translation by $f \in F$ lies in $[S]$, this shows that $R \subset S$ is a translation finite extension. $\square$

4.5. **Putting it all together.** We continue to use the setup from Section 4.2. Because $R \subset S$ is a translation finite extension (Lemma 4.5), we obtain

$$\operatorname{Cost}_\mu S \leq \operatorname{Cost}_\mu R + 1$$

from Lemma 2.11. In combination with Lemma 4.1, it follows that

$$\begin{aligned}\operatorname{Cost}_\mu \alpha = \operatorname{Cost}_\mu S &\leq \operatorname{Cost}_\mu R + 1 \\ &\leq \sum_{j=1}^{m} \mu(A_j) + 1 \leq |c|_1 + 1.\end{aligned}$$

Taking the infimum over all α-parametrised fundamental cycles c of M thus shows the desired estimate

$$\operatorname{Cost}_\mu \alpha - 1 \leq |M|^\alpha.$$

Because integral foliated simplicial volume can be computed in terms of ergodic essentially free parameter spaces [15, Proposition 4.17], taking the infimum over all ergodic essentially free standard Γ-spaces α implies that

$$\operatorname{Cost} \Gamma - 1 \leq |M|.$$

This completes the proof of Theorem 1.2.

4.6. **The weightless version.** The proof of the cost estimate of Theorem 1.2 does not incorporate the values of the coefficient functions. Therefore, the estimate can be improved in a straightforward way to the case of weightless parametrised simplicial volumes (Theorem 4.6). The advantage of these weightless versions is that they also allow for coefficients in finite fields and other commutative rings with unit [14].

We quickly review the definition of weightless parametrised simplicial volumes and indicate how to prove the theorem in this case. Let M be an oriented closed connected n-manifold with fundamental group Γ, let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a standard Γ-space, and let Z be a commutative ring with unit. We then write $L^\infty(X, Z) := Z \otimes_{\mathbb{Z}} L^\infty(X, \mathbb{Z})$ and

$$C_*(M; \alpha; Z) := L^\infty(X, Z) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; \mathbb{Z}).$$

A cycle $c \in C_*(M; \alpha; Z)$ is an $(\alpha; Z)$-fundamental cycle of M if c is homologous to a Z-fundamental cycle of M. Moreover, we define the *weightless norm* of a chain $c = \sum_{j=1}^m f_j \otimes \sigma_j \in C_n(M; \alpha; Z)$ in reduced form by

$$|c|_{(\alpha;Z)} := \sum_{j=1}^{m} \mu(\operatorname{supp} f_j) \in \mathbb{R}_{\geq 0}.$$

The *weightless parametrised Z-simplicial volume* of M is given by

$$|M|_{(\alpha;Z)} := \inf\bigl\{|c|_{(\alpha;Z)} \bigm| c \in C_n(M; \alpha; Z) \text{ is an } (\alpha; Z)\text{-fundamental cycle of } M\bigr\}.$$

Theorem 4.6. *Let M be an oriented closed connected manifold with fundamental group Γ, let $\alpha = \Gamma \curvearrowright (X, \mu)$ be an essentially free ergodic standard Γ-space, and let Z be a commutative ring with unit. Then*

$$\operatorname{Cost}_\mu \alpha - 1 \leq |M|_{(\alpha;Z)}.$$

It should be noted that as coefficient ring Z in Theorem 4.6 we can also take, e.g., finite fields. Therefore, we obtain an upper bound of cost in terms of objects in positive characteristic.

Proof. We can prove this version in the same way as the ℓ^1-version in Theorem 1.2. We will therefore only indicate the basic steps:

- As in the ℓ^1-case, we can assume without loss of generality that Γ is infinite. Let S be the orbit relation of α.
- Let $c = \sum_{j=1}^m f_j \otimes \sigma_j \in L^\infty(X,Z) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z})$ be an (α,Z)-fundamental cycle of M in reduced form.
- Literally in the same way as in the ℓ^1-case, we define the relation R on X associated with c.
- Then Lemma 4.1 shows that R is a subrelation of S and
$$\operatorname{Cost}_\mu R \leq |c|_{(\alpha;Z)}.$$
- For μ-almost every $x \in X$, the chain
$$c_x = \sum_{j=1}^m \sum_{\gamma\in\Gamma} f_j(\gamma^{-1}\cdot x)\cdot\gamma\cdot\sigma_j$$
is a well-defined locally finite Z-fundamental cycle in $C_n^{\mathrm{lf}}(\widetilde{M};Z)$ of $\widetilde{M}$ (the proof of the $\mathbb{Z}$-case [7, Lemma 2.5] also works for Z-coefficients).
- Using Lemma 4.4 and the arguments of the proof of Lemma 4.5, we obtain that $R \subset S$ is a translation finite extension.
- As in Section 4.5, we thus obtain $\operatorname{Cost}_\mu \alpha - 1 \leq |c|_{(\alpha;Z)} \leq |M|_{(\alpha;Z)}$, as claimed. $\square$

References

[1] M. Abért, N. Nikolov. Rank gradient, cost of groups and the rank versus Heegard genus problem, *J. Eur. Math. Soc.*, 14, 1657–1677, 2012. Cited on page: 1, 5

[2] S. Braun. *Simplicial Volume and Macroscopic Scalar Curvature*. PhD thesis, Karlsruhe Institute of Technology, 2018. Cited on page: 2, 8

[3] D. Fauser. Integral foliated simplicial volume and S^1-actions, preprint, available at arXiv:1704.08538 [math.GT], 2017. Cited on page: 2, 8

[4] D. Fauser, S. Friedl, C. Löh. Integral approximation of simplicial volume of graph manifolds, to appear in *Bull. Lond. Math. Soc.*, DOI 10.1112/blms.12266 Cited on page: 2, 8

[5] J. Feldman, C.C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. I. *Trans. Amer. Math. Soc.*, 234(2), 289–324, 1977. Cited on page: 4, 6

[6] S. Francaviglia, R. Frigerio, B. Martelli. Stable complexity and simplicial volume of manifolds, *J. Topol.*, 5(4), 977–1010, 2012. Cited on page: 3

[7] R. Frigerio, C. Löh, C. Pagliantini, R. Sauer. Integral foliated simplicial volume of aspherical manifolds, *Israel J. Math.*, 216(2), 707–751, 2016. Cited on page: 2, 3, 8, 10, 14

[8] D. Gaboriau. Coût des relations d'équivalence et des groupes, *Invent. Math.*, 139(1), 41–98, 2000. Cited on page: 2, 3, 5, 6, 7, 9

[9] D. Gaboriau. Invariants ℓ^2 de relations d'équivalence et de groupes, *Inst. Hautes Études Sci. Publ. Math.*, 95, 93–150, 2002. Cited on page: 3, 5

[10] M. Gromov. Volume and bounded cohomology, *Inst. Hautes Études Sci. Publ. Math.*, 56, 5–99, 1983. Cited on page: 2, 8

[11] A.S. Kechris, B.D. Miller. *Topics in Orbit Equivalence*, Springer Lecture Notes in Mathematics, vol. 1852, 2004. Cited on page: 2, 3, 5, 6, 7, 9

[12] C. Löh. Simplicial volume, *Bull. Man. Atl.*, 7–18, 2011. Cited on page: 8

[13] C. Löh. Rank gradient vs. stable integral simplicial volume, *Period. Math. Hung.*, 76(1), 88–94, 2018. Cited on page: 1

[14] C. Löh. Simplicial volume with $\mathbb{F}_p$-coefficients, to appear in *Period. Math. Hung.*, 2019. Cited on page: 13

[15] C. Löh, C. Pagliantini. Integral foliated simplicial volume of hyperbolic 3-manifolds, *Groups Geom. Dyn.*, 10(3), 825–865, 2016. Cited on page: 1, 2, 3, 8, 13

[16] W. Lück. Approximating L^2-invariants by their finite-dimensional analogues, *Geom. Funct. Anal.*, 4(4), 455–481, 1994. Cited on page: 3

[17] M. Schmidt. *L^2-Betti Numbers of $\mathcal{R}$-spaces and the Integral Foliated Simplicial Volume.* PhD thesis, Westfälische Wilhelms-Universität Münster, 2005. http://nbn-resolving.de/urn:nbn:de:hbz:6-05699458563 Cited on page: 2, 3, 8, 10

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